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A Modified Rumor Model on Infinite Cayley Trees

Jhon F. P. Tipas[1] Pablo M. Rodriguez[2]
Universidade Federal de Pernambuco, Recife, PE

Some of the earliest references to mathematical rumor models can be found in [1, 4]. The Maki-Thompson rumor model on a connected graph can be informally described as follows. The vertices represent individuals who can be classified into three categories: ignorants, spreaders, and stiflers. A spreader transmits the rumor to any of its nearest ignorant neighbors at a rate of one. At the same rate, a spreader becomes a stifler after contact with other nearest-neighbor spreaders or stiflers. In this work, we consider an extension of the Maki-Thompson rumor model on an infinite Cayley tree, assuming that as soon as an individual hears the rumor, they either spread it with probability $p \in (0,1]$ or remain neutral, becoming a stifler, with probability $1-p$. Of course, if we take $p = 1$ we recover the basic model. For a review of recent results on trees, we refer the reader to [2, 3]. We focus our attention on the infinite Cayley tree of coordination number $d+1$, with $d \geq 2$, $\mathbb{T} = \mathbb{T}_d$. The model is a continuous-time Markov process $(\eta_t)_{t\geq 0}$ with state space $\mathcal{S} = \{0,1,2\}^{\mathbb{T}}$. That is, at time t the state of the process is a function $\eta_t : \mathbb{T} \longrightarrow \{0,1,2\}$. We assume that each vertex $v \in \mathbb{T}$ represents an individual, and we say that such individual is an ignorant if $\eta(v) = 0$, a spreader if $\eta(v) = 1$, or a stifler if $\eta(v) = 2$. Moreover, if the system is in configuration $\eta \in \mathcal{S}$, the state of vertex v changes according to the following transition rates:

$$\begin{array}{cccc} & \text{transition} & & \text{rate} \\ 0 & \rightarrow & 1, & p\, n_1(v,\eta), \\ 0 & \rightarrow & 2, & (1-p)\, n_1(v,\eta), \\ 1 & \rightarrow & 2, & n_1(v,\eta) + n_2(v,\eta), \end{array} \tag{1}$$

where $n_i(v,\eta) = \sum_{u\sim v} 1\{\eta(u) = i\}$ is the number of nearest-neighbors of vertex v in state i for the configuration η, for $i \in \{1,2\}$. Formally, (1) means that if the vertex v is, say, in state 0 at time t then the probability that it will be in state 1 at time $t+h$, for small h, is $p\, n_1(x,\eta)h + o(h)$, where $o(h)$ represents a function such that $\lim_{h\to 0} o(h)/h = 0$. We call the Markov process $(\eta_t)_{t\geq 0}$ the Maki-Thompson rumor model on $\mathbb{T}$ with probability p of spreading, MT$(\mathbb{T}, p)$-model for short. In addition, we refer to the case when $\eta_0(\mathbf{0}) = 1$ and $\eta_0(v) = 0$ for all $v \neq \mathbf{0}$ as the *standard initial configuration.*

Definition 1. *Let $p \in (0,1]$ and consider the MT$(\mathbb{T}, p)$-model with the standard initial configuration. We say that the rumor propagates if, for any $t \geq 0$, there exists a vertex $v \in \mathbb{T}$ such that $\eta_t(v) = 1$. Otherwise, we say that the rumor becomes extinct.*

We denote the rumor propagation probability as $\theta(d,p)$ and we observe that Definition 1 is equivalent to [3, Definition 1]. It is not difficult to see, by a coupling argument, that $\theta(p,d)$ is non-decreasing as a function of p. That is, $\theta(p_1,d) \leq \theta(p_2,d)$, if $p_1 \leq p_2$. Therefore, we can define the critical parameter of the model as $p_c(d) := \inf\{p > 0 : \theta(p,d) > 0\}$.

[1]jhon.puerres@ufpe.br
[2]pablo@de.ufpe.br

Theorem 1. *Let $p \in (0,1], d \geq 3$, and consider the MT$(\mathbb{T}_d, p)$-model with the standard initial configuration. Then*

$$p_c(d) = \left\{ \frac{e^{d+1}\Gamma(d+1, d+1)}{(d+1)^d} - 1 \right\}^{-1}, \tag{2}$$

where $\Gamma(a,x)$ is the incomplete gamma function. Moreover, $p_c(d) \in (0,1)$ for any $d \geq 3$, and

$$p_c(d) \sim \sqrt{\frac{2}{\pi(d+1)}}. \tag{3}$$

Theorem 1 becomes more interesting when we realize that the MT$(\mathbb{T}_d, p)$-model exhibits a phase transition for any $d \geq 3$. We do not consider the case $d = 2$ because for this case [3, Theorem 1] guarantees that $p_c = 1$. For approximations of $p_c(d)$, see Table 1.

Table 1: Values of $p_c(d)$ for $d \in \{3, \ldots 11\}$.

d	3	4	5	6	7	8	9	10	11
p_c	0.8205	0.6620	0.5634	0.4955	0.4454	0.4067	0.3759	0.3505	0.3293

The main idea behind the proof of Theorem 1 is the identification of an underlying branching process related to the rumor model. After doing that we can apply well-known results of the theory of branching processes. This approach has been used before in [2, 3] to study rumor models on infinite random Cayley trees. For more details of the proofs of this result and applications for an extended model see [5].

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