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Finding Problems with Potential Quantum Supremacy using Graphs.

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The main motivation for developing quantum algorithms is their advantage over classical counterparts. In a query complexity model we wish to compute a boolean function $f : S \to T$, where $S \subseteq \Sigma^n$and T is a finite set [2]. It is possible to discover new problems with quantum advantage searching for second-degree linear polynomials, bounded between 0 and 1, which have a high L_1 norm [3]. This situation can be represented by defining an optimization problem, where L_1 is maximized for a proposed formulation of single-query quantum algorithms (which we denote as WDG). In this sense, in Theorem 1 an iterative method was presented, which produces a sequence of algorithms with increasing spectral normal L_1 from algorithms with spectral norm L_1 greater than 1. These contributions are detailed in the following paragraphs.

We say that a polynomial p approximates a function f with error bounded by ϵ, if $|p(x)-f(x)| < \epsilon$ for all x in the domain of f. Furthermore, we know that if there exist a $\epsilon < \frac{1}{2}$ and a $\epsilon' < \frac{1}{2}$, such that a partial boolean function f can be approximated by a $degree-2$ polynomial with error bounded by ϵ if and only if f is computable by a $1-query$ quantum algorithm with error bounded by ϵ' [1].

For each $b \in \{0,1\}^n$, we can define $\chi_b : \{1,-1\}^n \to \{-1,1\}$ such that $\chi_b(x) = \prod_i b_i x_i$. This family of functions is an orthonormal basis of the function space $f : \{1,-1\} \to \mathbb{R}$ [4].

Given $p : \{1,-1\}^n \to [0,1]$ a multilinear polynomial of degree at most two, then p has a unique representation

$$p = \sum_{b \in \{0,1\}^n} \alpha_b \chi_b, \tag{1}$$

such that $|b| \leq 2$.

Given this, we define a graph with no multiple edges $G(V,E)$, where V and E are the sets of vertices and edges respectively and denote $\omega : E \to \mathbb{R}$ and $x \in \{1,-1\}^{n+1}$ (*considering the ancilla bit* $x_0 = 1$), we call $D = (G, \omega)$ Weighted dynamical graph (**WDG**) and define

$$g_D(x) = \sum_{e \in E} s(e,x)\omega(e), \tag{2}$$

the value of D over x, where if $e = (v_i, v_j)$, then $s(e,x) = x_i x_j$ and if $e = (v_i)$, then $s(e,x) = x_0 x_i$. And associated to D we define a $(n+1) \times (n+1)$ real matrix M^D such that: *(i)* if $i = 0$ and $i \neq j$, then $M^D_{i,j} = \frac{1}{2}\omega(\{j\})$, *(ii)* if $j = 0$ and $i \neq j$, then $M^D_{i,j} = \frac{1}{2}\omega(\{i\})$, *(iii)* if $i = j$, then $M^D_{i,j} = 0$, and *(iv)* if $i, j > 0$, then $M^D_{i,j} = M^D_{j,i} = \frac{1}{2}\omega(\{i,j\})$. Where

$$g_D(x) = xM^D x^t. \tag{3}$$

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The optimization problem we discussed earlier is defined here as follows:

Definition 1. *Let f be a function such that $f : S \to [0,1]$ and $S \subset \{1,-1\}^n$. Considering a **WDG** $D = (G,\omega)$ and given some $\epsilon > 0$: Find C and ω that maximize*

$$\sum_{e \in E} |\omega(e)|. \tag{4}$$

subject to: (i) $|g_D(x) - f(x) + C| < \epsilon$ for each $x \in S$, and (ii) $\delta(D) = 1$. Where

$$\delta(D) = \max_{x \in \{-1,1\}^n} g_D(x) - \min_{x \in \{-1,1\}^n} g_D(x). \tag{5}$$

Where the equation 4 is the norm L_1 in the **WDG**. Now, consider the following terms:

- D and D' be WGDs such that $f(x) = (xM^D x^t + K)$ and $f'(y) = (yM^{D'} y^t + K')$ for $K, K' \in \mathbb{R}$, $S \subset \{1,-1\}^n$ and $T \subset \{1,-1\}^m$, where $f : S \to [0,1]$ and $f' : T \to [0,1]$ respectively.
- Denote $S^+ = \{x \in S : f(x) = 1\}$, $S^- = \{x \in S : f(x) = 0\}$, $T^+ = \{y \in T : f'(y) = 1\}$ and $T^- = \{y \in T : f'(y) = 0\}$.

if $L(f)$ be the fourier L_1 norm of f over its descomposition on functions χ_b, we state the following theorem

Theorem 1. *There is a **WDG** D'' such that i) $L(g_{D''}) = (L(g_D) + |K|)(L(g_{D'}) + |K'|) - |KK'|$ and ii) $g_{D''}(x) = f''(x) + K''$ for $x \in ((S^+ \otimes T^+) \cup (S^- \otimes T^-) \cup (S^+ \otimes T^-) \cup (S^- \otimes T^+))$ and some $K'' \in \mathbb{R}$. Where $f'' : \{1,-1\}^{nm} \to [0,1]$ satisfies (a) $f''(\omega) = 1$ if $\omega \in (S^+ \otimes T^+)$, and (b) $f''(\omega) = 0$ if $\omega \in ((S^+ \otimes T^-) \cup (S^- \otimes T^+) \cup (S^- \otimes T^-))$.*

Here we show a relatively simple framework for problem search where single-query quantum algorithms can have potential advantage over classical decision trees. A possible strategy for developing quantum algorithms is to first solve the optimization problem using generic optimization methods, in order to generalize the algorithm through a sequence of **WGDs** obtained by applying the Theorem 1.

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