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Control Strategies for the McKean-Vlasov Equation

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We study the non-linear McKean-Vlasov equation

$$\partial_t \mu = \sigma \Delta \mu + \nabla \cdot [\mu(\nabla V + \nabla W * \mu)], \quad \mu(x,0) = \mu_0(x), \tag{1}$$

on a domain $\Omega \subset \mathbb{R}^d$ with periodic boundary conditions, where $\sigma > 0$ is the diffusion parameter, V is a confining potential, and W is an interaction potential. This equation arises as the mean-field limit of the interacting particle system

$$dx_t^i = -\nabla V(x_t^i)\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla W(x_t^i - x_t^j)\,dt + \sqrt{2\sigma}\,dB_t^i, \tag{2}$$

as $N \to \infty$ (see, e.g., [5]). Applications include opinion dynamics, synchronization [3], and collective motion [2]. Under the convexity of W and the uniform convexity of V, the existence and uniqueness of a stationary state $\bar{\mu}$ and exponential convergence (via a logarithmic Sobolev inequality [4]) are well established. Even under convexity, a small spectral gap can lead to slow convergence, motivating acceleration. In the absence of convexity, multiple steady states may arise, and one may wish to guide the dynamics toward a particular equilibrium.

To accelerate convergence toward a desired steady state $\bar{\mu}$, we introduce control functions

$$V(x) \mapsto V(x) + \sum_{j=1}^{m} \alpha_j(x) u_j(t), \tag{3}$$

where $\alpha_j(x)$ are spatial shape functions and $u_j(t)$ are time-dependent control inputs. Our objective is to design α_j and u_j so that the distribution μ converges to $\bar{\mu}$ as fast as possible. Setting $y = \mu - \bar{\mu}$,

$$\dot{y} = \mathcal{A}y + \mathcal{W}(y + \bar{\mu}) - \mathcal{W}(\bar{\mu}) + \sum_{j=1}^{m} u_j(t)\mathcal{N}_j y + \sum_{j=1}^{m} u_j(t)\mathcal{N}_j \bar{\mu} \tag{4}$$

is the controlled McKean-Vlasov equation in abstract form with $\mathcal{A}\mu := \nabla \cdot (\mu \nabla V + \sigma \nabla \mu)$, $\mathcal{N}_j \mu := \nabla \cdot (\mu \nabla \alpha_j)$, and $\mathcal{W}(\mu) := \nabla \cdot (\mu \nabla W * \mu)$. Following the approach in [1] for the (linear) Fokker-Planck equation, we linearize the nonlinear operator $\mathcal{W}$ about $\bar{\mu}$ via its Fréchet derivative, i.e., $D\mathcal{W}(\bar{\mu})[y] \approx \mathcal{W}(y + \bar{\mu}) - \mathcal{W}(\bar{\mu})$. We then consider the quadratic cost functional

$$J(u) = \frac{1}{2}\int_0^{\infty} \left(\langle y, \mathcal{M}y \rangle_{L^2(\Omega)} + \|u(t)\|_2^2 \right) dt. \tag{5}$$

By solving the associated Riccati equation for the linearized system, we derive an optimal feedback law that drives the solution of (1) to $\bar{\mu}$. The spatial control functions α_j are fixed by selecting them from the Fourier basis.

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For numerical implementation, we employ a spectral method based on Fourier basis functions, which is natural on the torus. Specifically, we approximate $y(x,t) \approx \hat{y}_L(x,t) = \sum_{k=-L}^{L} a_k(t)\phi_k(x)$, where $\{\phi_k\}$ denotes the Fourier basis. This projection yields a finite-dimensional system in terms of the Fourier coefficients $a_k(t)$. In this finite-dimensional setting, the optimal control problem leads to the matrix Riccati equation

$$A^*\hat{\Pi} + \hat{\Pi}A - \hat{\Pi}BB^*\hat{\Pi} + M = 0, \tag{6}$$

which provides the optimal feedback law $u(t) = -B^*\hat{\Pi}a(t)$, where $a(t) = (a_{-L}(t), \dots, a_L(t))^\top$. This feedback is then incorporated into the discretized non-linear dynamics, forming a closed-loop dynamics.

Figure 1 shows the results of two numerical experiments, demonstrating that the proposed control strategy effectively accelerates convergence toward the chosen steady state. We consider the Kuramoto model [3] with $V = 0$ and $W(x) = -K\cos(x)$ with four controls. For $K \leq 1$, the uniform distribution is the unique and stable steady state. In contrast, for $K > 1$, the uniform state becomes unstable, and non-uniform steady states emerge, but by using control, we are able to steer the dynamics towards the uniform distribution.

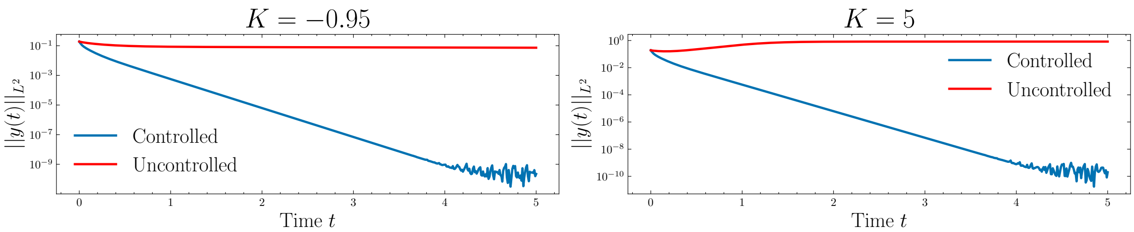

Figure 1: Evolution of the L^2 norm of $y(t)$ for controlled (blue) and uncontrolled (red) cases where $\bar{\mu}$ is the uniform distribution. Left: $K = -0.95$; right: $K = 5$. Source: authors.

Looking ahead, our future work will focus on a detailed error analysis of the spectral discretization, an exploration of the relationship between the control shape functions and the spectral gap, and extensions to more general interaction potentials and domain geometries. Additionally, we will provide theoretical results concerning the stability of the optimal control problem.

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