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Counting Cycles in Strongly Connected Graphs

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Resumo: *In this short presentation I give a formula for counting cycles of a given length in strongly connected graphs and some of its properties.*

Palavras-chave: *Counting, Cycles , Strongly Connected Graphs*

1 Introduction

Let N be a positive integer, R a real number, μ the classical Möbius function defined by the rules: a) $\mu(+1) = +1$, b) $\mu(g) = 0$, if $g = p_1^{e_1}...p_q^{e_q}$, $p_1, ..., p_q$ primes, and any $e_i > 1$, c) $\mu(p_1...p_q) = (-1)^q$. The polynomial of degree N in R with rational coefficients given in terms of Möbius function,

$$\mathcal{M}(N; R) = \frac{1}{N} \sum_{g|N} \mu(g) R^{\frac{N}{g}}, \tag{1.1}$$

is called the *Witt polynomial*. It satisfies the formal relation

$$\prod_{N=1}^{\infty} (1 - z^N)^{\mathcal{M}(N;R)} = 1 - Rz \tag{1.2}$$

called the *Witt identity*. See the introduction to [4] for a nice account about these relations.

In [5] (see [1], section 4, for a proof) Sherman remarks that $\mathcal{M}(N; R)$ is the number of equivalence classes of non-periodic cycles of length N which traverse counterclockwisely the edges of a graph with R loops counterclockwisely oriented and hooked to a single vertex. With this result in mind a natural question to ask is whether (1) and (2) can be generalized to other graphs. The question has a positive answer. The objective here is to give the generalizations of (1) and (2) for strongly connected graphs. In section 2, the basic formula for counting cycles is given. In section 3, some properties of the copunting formula are given.

2 Counting cycles of a given length

The results in this section are based on ideas from [2] and [6].

Let $G = (V, E)$ be a *finite directed and strongly connected graph* where V is the set of vertices with $|V|$ elements and E is the set of oriented edges with $|E|$ elements labeled e_1, ...,$e_{|E|}$. An edge has an origin and an end as given by its orientation. The graph may have multiple edges and loops but no 1-degree vertices. A graph is strongly connected if it contains a directed path from a to b and a directed path from b to a for every pair of vertices a, b.

A path in G is given by an ordered sequence $(e_{i_1}, ..., e_{i_N})$, $i_k \in \{1, ..., |E|\}$, of oriented edges such that the end of e_{i_k} is the origin of $e_{i_{k+1}}$. The orientation of an edge defines the origin and the end of it.

All paths considered here are cycles. These are closed paths, that is, the end of e_{i_N} coincides with the origin of e_{i_1}. A cycle has a natural orientations induced by the orientation of its edges. The length of a cycle is the number of edges in its sequence. A cycle p is called periodic if $p = q^r$ for some $r > 1$ and q is a non periodic cycle. Number r is called the period of p. The sequence $(e_{i_N}, e_{i_1}, ..., e_{i_{N-1}})$ is called a circular permutation of $(e_{i_1}, ..., e_{i_N})$. Circular permutations of a sequence represent the same path. The inverse of a path is not possible.

The classical Möbius inversion formula is used here. Given arithmetic functions f and g it states that $g(n) = \sum_{d|n} f(d)$ if and only if $f(n) = \sum_{d|n} \mu(d) g(n/d)$.

In order to count cycles of a given length in a graph G a crucial tool is the edge adjacency matrix of G. This is the $|E| \times |E|$ matrix T defined as follows: $T_{ij} = 1$, if end vertex of edge i is the start vertex of edge j; $T_{ij} = 0$, otherwise.

Theorem 2.1. *The number TrT^N (over)counts cycles of length N in a graph G.*

Proof. Let a and b be two edges of G. The $(a, b)^{th}$ entry of matrix T^N is

$$(T^N)_{(a,b)} = \sum_{e_{i_1},...,e_{i_{N-1}}} T_{(a,e_{i_1})} T_{(e_{i_2},e_{i_3})} ... T_{(e_{i_{N-1}},b)}$$

From the definition of the entries of T it follows that $(T^N)_{(a,b)}$ counts the number of paths of length N from edge a to edge b. For $b = a$, only cycles are counted. Taking the trace gives the number of cycles with every edge taken into account as starting edge, hence, the trace overcounts cycles because every edge in a cycle is taken into account as starting edge. □.

Theorem 2.2. *Denote by $\alpha(N, T)$ the number of equivalence classes of non periodic cycles of length N which traverse a graph G. This number is given by the following formula:*

$$\alpha(N, T) = \frac{1}{N} \sum_{g|N} \mu(g) \, TrT^{\frac{N}{g}} \tag{2.1}$$

Proof. In the set of TrT^N cycles there is the subset with $N\alpha(N, T)$ elements formed by the non periodic cycles of length N plus their circular permutations and the subset with $\sum_{g \neq 1|N} \frac{N}{g} \alpha(\frac{N}{g}, T)$ elements formed by the periodic cycles of length N (whose periods are the common divisors of N) plus their circular permutations. (A cycle of period g and length N is of the form

$$(e_{k_1} e_{k_2} ... e_{k_t})^g$$

where $t = N/g$, and $(e_{k_1} e_{k_2} ... e_{k_t})$ is a non periodic cycle so that the number of periodic cycles with period g plus their circular permutations is given by $(N/g)\alpha(N/g, T))$. Hence,

$$TrT^N = \sum_{g|N} \frac{N}{g} \, \alpha\left(\frac{N}{g}, T\right)$$

Möbius inversion formula gives the result. □

Remark 1. Some terms in the right hand side of (2.1) are negative. In spite of that the right hand side is always positive. Multiply both sides by N. The fist term equals TrT^N while the other terms give (in absolute value) the number according to period of the various subsets of periodic cycles which are proper subsets of the larger set with TrT^N elements.

Remark 2. (1.1) is a special case of (2.1). Let G be the graph In this case T is the $R \times R$ matrix with all entries equal to one and $TrT^N = R^N$ so that $\alpha(N, T) = \mathcal{M}(N; R)$.

Theorem 2.3. *Define*

$$g(z) := \sum_{N=1}^{\infty} \frac{TrT^N}{N} z^N. \tag{2.2}$$

Then,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\alpha(N)} = e^{-g(z)} = det(1 - zT) \tag{2.3}$$

Proof. The result follows taking the formal logarithm of both sides of equalities. □

Remark 3. (1.2) is a special case of (2.3) when G is the graph with R loops counterclockwise oriented and hooked to a single vertex. In this case $det(1 - zT) = 1 - Rz$ where T is the matrix with all entries equal to 1. See Remark 2.

Remark 4. The identity (2.3) have a remarkable resemblance with the Ihara-Bass identity. They are not the same objects because in the latter the underlying graph need not be strongly connected and cycles can traverse an edge following the inverse orientation. See [2] and [6].

Remark 5. It is well known that the number of cycles of length N in a graph G can be computed using the vertex adjacency matrix A, hence, $TrT^N = TrA^N$. The reciprocal of (2.3) with the matrix T replaced by A is the so called Bowen-Lanford zeta function of a graph. The reciprocal of (2.3) is this function as expressed in term of the edge adjacency matrix.

Example. G_1, the graph shown in the Figure. The edge matrix of G_1 is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The matrix has the trace $TrT^N = 0$ if N is odd and $TrT^N = 2^{\frac{N}{2}+1}$ if N is even, and the determinant

$$det(1 - zT) = 1 - 2z^2.$$

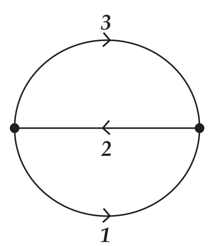

Figure 1: Graph G_1

The number of classes of nonperiodic cycles of length N is $\alpha(N) = 0$, if N is odd; for N even,

$$\alpha(N) = \frac{1}{N} \sum_{\substack{g|N \\ N/g \text{ even}}} \mu(g) 2^{\frac{N}{2g}+1}.$$

The first few values are $\alpha(2) = 2$, $\alpha(4) = 1$, $\alpha(6) = 2$, $\alpha(8) = 3$, $\alpha(10) = 6$. For $N = 2$, the classes are $[e_1 e_2]$ and $[e_2 e_3]$. For $N = 4$, only $[e_1 e_2 e_3 e_2]$. Furthermore,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\alpha(N)} = 1 - 2z^2.$$

3 Some properties

In [7] Metropolis and Rota proved that the Witt polynomials satisfy several important identities which they used to build the necklace algebra. Theorem 3.1 shows that α satisfies similar identities. Theorem 3.2 gives a generalization of the classical *Strehl identity* [8]. Theorem 3.3 below together with results from [3] imply that (4) and (5) can be interpreted as data associated to a Lie superalgebra.

Theorem 3.1. *Given the matrices T_1 and T_2 denote by $T_1 \otimes T_2$ the Kronecker product of T_1 and T_2. Then,*

$$\sum_{[s,t]=N} (s,t)\alpha(s,T_1)\alpha(t,T_2) = \alpha(N, T_1 \otimes T_2), \tag{3.1}$$

(s,t) is the maximum common divisor of s and t. The summation is over the set of all positive integers s,t such that $[s,t] = N$, $[s,t]$ the least common multiple of s,t. Also,

$$\alpha(N, T^l) = \sum_{[l,t]=Nl} \frac{t}{N}\alpha(t,T). \tag{3.2}$$

and

$$(r,s)\alpha(N, T_1^{s/(r,s)} \otimes T_2^{r/(r,s)}) = \sum (rp, sq)\alpha(p,T_1)\alpha(q,T_2) \tag{3.3}$$

The sum is over p,q such that $pq/(pr,qs) = N/(r,s)$.

Theorem 3.2.

$$\prod_{k\geq 1}\left[\frac{1}{det(1-z^kT_1)}\right]^{\alpha(k,T_2)} = \prod_{j\geq 1}\left[\frac{1}{det(1-z^jT_2)}\right]^{\alpha(j,T_1)} \tag{3.4}$$

Theorem 3.3. *Define $g(z) := \sum_{N=1}^{\infty} \frac{TrT^N}{N} z^N$. Then,*

$$\prod_{N=1}^{+\infty} (1-z^N)^{\pm\Omega(N,T)} = e^{\mp g(z)} = [det(1-zT)]^{\pm} = 1 \mp \sum_{i=1}^{+\infty} c_{\pm}(i) z^i, \tag{3.5}$$

where

$$c_{\pm}(i) = \sum_{m=1}^{i} \lambda_{\pm}(m) \sum_{\substack{a_1+2a_2+\ldots+ia_i=i \\ a_1+\ldots+a_i=m}} \prod_{k=1}^{i} \frac{(TrT^k)^{a_k}}{a_k! k^{a_k}} \tag{3.6}$$

with $\lambda_+(m) = (-1)^{m+1}$, $\lambda_-(m) = +1$, $c_+(i) = 0$ for $i > 2|E|$, and $c_-(i) \geq 0$. Furthemore,

$$TrT^N = N \sum_{s\in S(N)} (\pm 1)^{|s|+1} \frac{(\mid s \mid -1)!}{s!} \prod c_{\pm}(i)^{s_i} \tag{3.7}$$

where $S(N) = \{s = (s_i)_{i\geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum is_i = N\}$ and $\mid s \mid = \sum s_i$, $s! = \prod s_i!$.

In section 2.3 of [3], given a formal power series $\sum_{i=1}^{+\infty} t_i z^i$ with $t_i \in \mathbb{Z}$, for all $i \geq 1$, the coefficients in the series are interpreted as the superdimensions of a $\mathbb{Z}_{>0}$-graded superspace $V = \bigoplus_{i=1}^{\infty} V_i$ with dimensions $dimV_i = |t_i|$ and superdimensions $DimV_i = t_i \in \mathbb{Z}$. Let $\mathcal{L}$ be the free Lie superalgebra generated by V. Then, $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ and the subspaces $\mathcal{L}_N$ have dimension given by

$$Dim\mathcal{L}_N = \sum_{g|N} \frac{\mu(g)}{g} W\left(\frac{N}{g}\right) \tag{3.8}$$

where g ranges over all common divisors of N,

$$W(N) = \sum_{s \in S(N)} \frac{(\mid s \mid -1)!}{s!} \prod t(i)^{s_i} \tag{3.9}$$

is called the *Witt partition function.* Furthermore,

$$\prod_{N=1}^{\infty} (1 - z^N)^{\pm Dim \mathcal{L}_N} = 1 \mp \sum_{i=1}^{\infty} f_{\pm}(i) z^i. \tag{3.10}$$

where $f_+(i) = t(i)$ and $f_-(i) = dimU(\mathcal{L})_i$ is the dimension of the i-th homogeneous subspace of the universal enveloping algebra $U(\mathcal{L})$.

Apply this interpretation to the determinant (2.3) which is a polynomial of degree $|E|$ in the formal variable z. It can be taken as a power series with coefficients $t_i = 0$, for $i > |E|$. Comparison of (3.8), (3.9), (3.10) with (3.5), (3.6), (3.7) implies the following result:

Theorem 3.4. *Given a graph G, T its edge adjacency matrix, let $V = \bigoplus_{i=1}^{|E|} V_i$ be a $\mathbb{Z}_{>0}$-graded superspace with finite dimensions $dimV_i = |c_+(i)|$ and the superdimensions $DimV_i = c_+(i)$ given by the coefficients of $det(1 - zT)$ (see (3.6)). Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free Lie superalgebra generated by V. Then, $\mathcal{L}_N$ has superdimension $Dim\mathcal{L}_N = \alpha(N)$. The algebra has generalized Witt identity given by (2.3) and its reciprocal is the generating function for the dimensions of the subspaces of the enveloping algebra $U(\mathcal{L})$ which are given by $c_-(i)$ in (3.6).*

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