A Virtual Element Numerical Approximation for the Vibration Problem of a Thin Plate

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Abstract. We focus our attention in the development of a virtual element method for the approximation of the vibration problem of a thin plate modeled by Kirchhoff-Love equations. We introduce a weak variational formulation based on the Sobolev space H^2 . In addition, we propose a discretization by means of the lowest-order non conforming elements. We show that the resulting scheme provides a correct approximation of the spectrum and prove optimal-order error estimates. Finally, we report some numerical tests supporting our theoretical results.

Keywords. Vibration Plate, Virtual Element Method, Clamped Plate, Simply Supported, Numerical Results

1 Introduction

In this work we study the analysis of a *virtual element method* (VEM) [3] to compute the vibration modes of an elastic plate. This problem has attracted much interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the vibration frequencies and modes of the plate (i.e., eigenvalues-vibration frequencies and eigenfunctions-vibration modes).

In this work we propose and analyze a virtual element method for the vibration problem of a plate modeled with the Kirchhoff-Love equations [4]. From the associated source problem, a solution operator is defined, which allows to characterize the spectrum of the vibration plate with its spectrum. In order to approximate the eigenvalues and eigenfunctions associated with the vibration problem, a discrete version of the solution operator is defined. Estimates of the error and order of convergence are obtained for the eigenfunctions and a double order for the eigenvalues.

The outline of this work is as follows. In Section 2 we present the equations that describe the vibration problem of a thin plate. We propose a weak variational formulation associated to the model problem. Using the Lax-Milgram theory for the associated source problem of said formulation, a solution operator is defined that characterizes the continuous spectrum with its spectrum. In order of convergence is obtained for the eigenfunctions and double order for the eigenvalues independent on the parameter h of the discretization. We present some numerical tests in Section 4 with configuration domains for the plate in L-shaped and square-shaped domains. We illustrate through graphs and tables these numerical results where the double order for the eigenvalues predicted in the theoretical analysis can be verified.

Throughout the document we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes, or hats a generic constant independent of the mesh size h, which may take different values in different occurrences. Let $\mathcal{O} \subset \Omega$ an open set, and $s \in [0, \infty)$. We will indicate the standard Sobolev seminorms (also for the norms) with the following shorter symbols $|v|_s = |v|_{H^s(\mathcal{O})}$, $|v|_s = |v|_{H^s(\mathcal{O})}$ for all $v \in H^s(\mathcal{O})$. Moreover, we denote by $\mathbb{P}_k(\mathcal{O})$ the set of polynomial of degree up to k defined on $\mathcal{O}$.

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DOI: 10.5540/03.2026.012.01.0241

2 The Model Problem

The vibration eigenvalue problem of a simply supported plate **(VEP-SSP)** and The vibration eigenvalue problem of a clamped plate **(VEP-CP)** modeled by the Kirchhoff-Love equations, are formulated as follows (see for instance, [4, 8]): Find (λ, u), with $u \neq 0$, such that

$$\left\{\begin{array}{ll}\textbf{VEP-SSP} & \\ \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Gamma,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll}\textbf{VEP-CP} & \\ \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \partial_{\boldsymbol{n}} u = 0 & \text{on } \Gamma,\end{array}\right. \tag{1}$$

where $\Gamma := \partial\Omega$. In both systems of equations, the parameter $\lambda = \omega^2$, with $\omega > 0$ representing the vibrational frequency.

2.1 The Continuous Variational Formulation

We have that the weak formulations of the spectral problems (1) are given by. Find $(\lambda, u) \in \mathbb{R} \times \mathrm{H}^{\dagger}$, with $u \neq 0$, such that

$$a(u, v) := \int_{\Omega} \nabla^2 u : \nabla^2 v = \lambda \int_{\Omega} u\, v =: \lambda b(u, v) \qquad \forall v \in \mathrm{H}^{\dagger}, \tag{2}$$

where, $\nabla^2 u$ denotes the Hessian matrix of u, and we have employed the superscript $\dagger \in \{\mathbf{SSP}, \mathbf{CP}\}$ to refer us to the following Sobolev spaces:

$$H^{\mathbf{SSP}} := \Big\{ v \in H^2(\Omega) : v = 0 \text{ on } \Gamma \Big\} \qquad \text{and} \qquad H^{\mathbf{CP}} := \Big\{ v \in H^2(\Omega) : v = \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma \Big\}.$$

Now, in order to characterize the spectrum of the continuous spectral problem (2) we introduce the following solution operator. Given any f belonging to $L^2(\Omega)$ define the operator $T : L^2(\Omega) \to H^{\dagger} \subset L^2(\Omega)$ by $Tf =: \widetilde{u}$ where $\widetilde{u}$ is the unique solution of the source problem $a(\widetilde{u}, v) = b(f, v) \quad \forall v \in H^{\dagger}$. It is easy to check that Lax-Milgram Theorem guarantees the well definition and boundedness of the operator T. Moreover, (λ, u) solves the spectral problem (2) if and only if $Tu = \mu u$, where $\mu := \frac{1}{\lambda} \neq 0$ and $u \neq 0$. In addition, we have that T is a compact and self-adjoint operator. Therefore, the spectrum of operator T is a sequence of real positive eigenvalues that converges to 0. Moreover, the multiplicity of each eigenvalue is finite (see [4]).

3 The Discrete Variational Formulation

Let Ω_h be a discretization of Ω composed of generic polyhedrons K. For all $K \in \Omega_h$, we denote by e a general edge in ∂K. In addition, we denote by $\mathcal{E}_h$ and $\mathbf{v}$ the set of all edges in Ω_h and a generic vertice on $\mathcal{E}_h$, respectively.

The non-conforming VEM scheme of the eigenvalue problem (2), read as: seek non-zero $(u_h, \lambda_h) \in V_h \times \mathbb{R}$, such that

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in V_h, \tag{4}$$

where space V_h is established in [2, 9] and defined as follows:

$$V_h := \left\{ v_h \in H^{2,\mathrm{NC}}(\Omega_h) : v_h|_K \in V_h(K) \quad \forall K \in \Omega_h \right\}, \tag{5}$$

$$H^{2,\mathrm{NC}}(\Omega_h) := \Big\{ v_h \in L^2(\Omega) : v_h|_K \in H^2(K)\ \forall K \in \Omega_h; v_h(\mathbf{v}) = 0\ \forall \mathbf{v} \in \partial\Omega; \\ ([\![\partial_{\mathbf{n}_e} v_h]\!], q)_{0,e} = 0\ \forall q \in \mathbb{P}_0(e)\ \forall e \in \mathcal{E}_h \Big\}, \tag{6}$$

$$V_h(K) := \Big\{ v_h \in H^2(K) : \Delta^2 v_h \in \mathbb{P}_2(K),\ \ v_h|_e \in \mathbb{P}_2(e),\ \ \Delta v_h|_e \in \mathbb{P}_0(e) \quad \forall e \in \partial K$$
$$(q, v_h - \Pi_K^{\Delta} v_h)_{0,K} = 0 \quad \forall q \in \mathbb{P}_2(K),\ \ (q, v_h - \Pi_K^{\Delta} v_h)_{0,e} = 0\ \ \forall e \subset \partial K \Big\}. \tag{7}$$

In addition, $H^{2,\mathrm{NC}}(\Omega_h)$ is endowed with the broken seminorm (which is a norm on V_h) by:

$$|v_h|_{2,h} := \Big(\sum_{K \in \Omega_h} |v_h|_{2,K}^2 \Big)^{1/2}.$$

Now, in order to introduce the discrete solution operator we first establish the following result, whose proof can be obtained by following the same arguments as for [5, Lemma 4.1]

Lemma 3.1. *There exist positive constants C_Ω and C_P such that*

$$\begin{aligned} |a_h(u_h, v_h)| &\leq C_a ||u_h||_{2,h} ||v_h||_{2,h} && \forall u_h, v_h \in V_h \\ |b_h(u_h, v_h)| &\leq C_b ||u_h||_{0,\Omega} ||v_h||_{0,\Omega} && \forall u_h, v_h \in V_h, \\ a_h(v_h, v_h) &\geq C_P ||v_h||_{2,\Omega}^2 && \forall v \in V_h. \end{aligned}$$

Given any f_h belonging to V_h define the operator $T_h : V_h \to V_h \subset L^2(\Omega)$ by $T_h f_h =: \widetilde{u}_h$ where $\widetilde{u}_h$ is the unique solution of the source problem $a_h(\widetilde{u}_h, v_h) = b_h(f_h, v_h) \qquad \forall v_h \in V_h$.

It is easy to check that Lax-Milgram and Lemma 3.1 guarantee the well definition and boundedness of the operator T_h. Moreover, resorting to the standard spectral theory for compact operators (see for instance [4]) we obtain the following results, which establishes the convergence of the discrete spectrum to the continuous one.

Theorem 3.1. *There exist $s > 1/2$ and $C > 0$ independent of h such that*

$$\begin{aligned} \|(T - T_h)v\|_{0,\Omega} &\leq C h^{2s} ||v||_{0,\Omega}, \\ |\lambda - \lambda_h| &\leq C h^2, \end{aligned} \tag{8a}$$

for all eigenfunction $v \in H^{2+s}(\Omega)$ with $s \geq 1/2$.

The proof of the Teorem 3.1 can be obtained with the same arguments as those applied in [1, Theorem 5.2].

Remark 3.1. *The constant s in the lemma above represents the Sobolev regularity associated with the biharmonic equation under homogeneous Dirichlet boundary conditions. This constant depends solely on the geometry of the domain Ω. If Ω is convex, then $s = 1$. Otherwise, the Lemma holds for all $s < s_0$, where $s_0 \in \left(\frac{1}{2}, 1\right)$ is determined by the largest reentrant angle of Ω (see [6] for the exact expression defining s_0).*

4 Numerical Results

In this section we employ a Matlab code to approximate the lowest eigenvalues of the vibration problem of a thin plate through the discrete scheme analyzed in this work. We have computed the eigenvalues considering as parameter $h = \frac{1}{\sqrt{N_P}}$ for $N_P = 32^2, 64^2, 128^2$, where N_P denotes the number of polygons inside the mesh Ω_h. In addition, we have calculated the rate of convergence for the first four lowest eigenvalues with a least-squares fit of the form

$$\lambda(h) \approx \lambda_{ih}^E + h^{\alpha_i}$$

where α_i is the rate of convergence approximated of each extrapolated eigenvalue λ_{ih}^E , with $i \in \mathbb{N}$ (see for instance [7]). We consider the following configurations for the domain Ω.

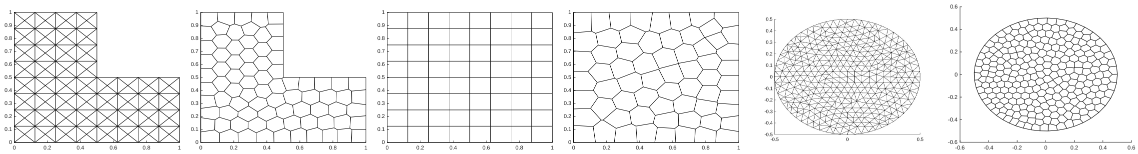

Figure 1: Meshes (from left to right): $\Omega_{\mathrm{L}}^{\mathrm{t}}$,$\Omega_{\mathrm{L}}^{\mathrm{v}}$, $\Omega_{\mathrm{S}}^{\mathrm{s}}$,$\Omega_{\mathrm{S}}^{\mathrm{v}}$,$\Omega_{\mathrm{C}}^{\mathrm{t}}$, $\Omega_{\mathrm{C}}^{\mathrm{v}}$. Source: Figure created by the author

4.1 Test: Non-Convex Domain

In this test, we consider the following domain $\Omega_L := (0,1)^2 - \{[1/2,1] \times [1/2,1]\}$ to compute the the first four lowest eigenvalues associated to **VEP-CP** and we report them in Table 1. In this case, the method converges with an order close to 1.44 for the first eigenvalue, which is as expected due to the singularity of the solution (cf. Remark 3.1). However, for the second, third, and fourth eigenvalues, the method achieves higher convergence orders. In addition Figure 2 illustrates the four eigenfunctions $u_{1h}, u_{2h}, u_{3h}, u_{4h}$ of a L-shaped plate for **VEP-CP**.

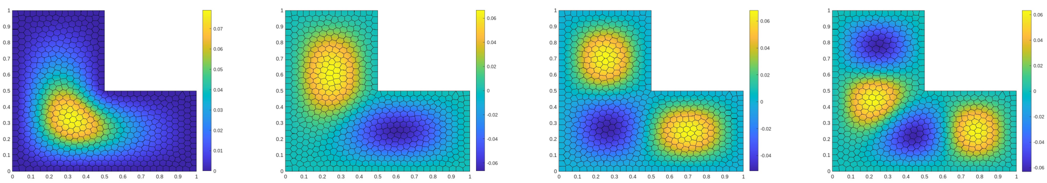

Figure 2: Eigenfunctions $u_{1h}, u_{2h}, u_{3h}, u_{4h}$. Source: Figure created by the author

Table 1: Lowest eigenvalues of a L-shaped plate for **VEP-CP**.

$\Omega_{\mathrm{L}}^{\mathrm{t}}$	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 3072$	6.4753e+03	1.0810e+00	1.4547e+00	2.5414e+00
$N_P = 12288$	6.6159e+03	1.0990e+00	1.4813e+00	2.5961e+00
$N_P = 49152$	6.6676e+03	1.1038e+00	1.4882e+00	2.6104e+00
rate of conv.	1.4400e+00	1.9100e+00	1.9500e+00	1.9300e+00
λ_{ih}^E	6.6978e+03	1.1055e+00	1.4906e+00	2.6155e+00
$\Omega_{\mathrm{L}}^{\mathrm{v}}$	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 640$	6.3858e+03	1.0684e+04	1.4422e+04	2.5179e+04
$N_P = 2560$	6.5861e+03	1.0956e+04	1.4782e+04	2.5901e+04
$N_P = 10240$	6.6276e+03	1.1038e+00	1.4882e+00	2.6204e+00
rate of conv.	1.4400e+00	1.9100e+00	1.9500e+00	1.9300e+00
λ_{ih}^E	6.6970e+03	1.1057e+04	1.4905e+04	2.6153e+04

4.2 Test: Square-Shaped Domain

In this case, we consider $\Omega_{\mathrm{S}} := (0,1) \times (0,1)$ to compute the first four lowest eigenvalues associated with **VEP-SSP** and **VEP-CP**. These values are reported in Tables 2 and 3, respectively.

In both cases, the order of convergence for the eigenvalues is expected to be equal to 2 (cf. Theorem 3.1). In addition, Figure 3 illustrates the four eigenfunctions associated with the first four lowest vibration eigenvalues of a square-shaped simply supported plate.

Table 2: Lowest eigenvalues of a square-shaped plate for **VEP-SSP** B.C.

Ω_S^s	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 1024$	3.8860e+02	2.4192e+03	2.4192e+03	6.1690e+03
$N_P = 4096$	3.8938e+02	2.4312e+03	2.4312e+03	6.2176e+03
$N_P = 16384$	3.8957e+02	2.4342e+03	2.4342e+03	6.2300e+03
rate of conv.	1.9900e+00	1.9800e+00	1.9800e+00	1.9700e+00
exact value	$4\pi^4$	$25\pi^4$	$25\pi^4$	$64\pi^4$
Ω_S^v	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 1024$	3.8868e+02	2.4196e+03	2.4207e+03	6.1724e+03
$N_P = 4096$	3.8939e+02	2.4314e+03	2.4315e+03	6.2188e+03
$N_P = 16384$	3.8958e+02	2.4343e+03	2.4343e+03	6.2303e+03
rate of conv.	1.9700e+00	2.0300e+00	1.9200e+00	2.0100e+00
exact value	$4\pi^4$	$25\pi^4$	$25\pi^4$	$64\pi^4$

Table 3: Lowest eigenvalues of a square-shaped plate for **VEP-CP**.

Ω_S^s	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 1024$	1.2817e+03	5.2928e+03	5.2928e+03	1.1427e+00
$N_P = 4096$	1.2916e+03	5.3627e+03	5.3627e+03	1.1638e+00
$N_P = 16384$	1.2941e+03	5.3806e+03	5.3806e+03	1.1692e+00
rate of conv.	1.9800e+00	1.9600e+00	1.9600e+00	1.9400e+00
λ_{ih}^E	1.2949e+03	5.3869e+03	5.3869e+03	1.1712e+00
Ω_S^v	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 1024$	1.2830e+03	5.2963e+03	5.3044e+03	1.1442e+04
$N_P = 4096$	1.2918e+03	5.3638e+03	5.3646e+03	1.1642e+04
$N_P = 16384$	1.2942e+03	5.3811e+03	5.3811e+03	1.1694e+04
rate of conv.	1.9200e+00	1.9700e+00	1.8600e+00	1.9600e+00
λ_{ih}^E	1.2950e+03	5.3870e+03	5.3875e+03	1.1712e+04

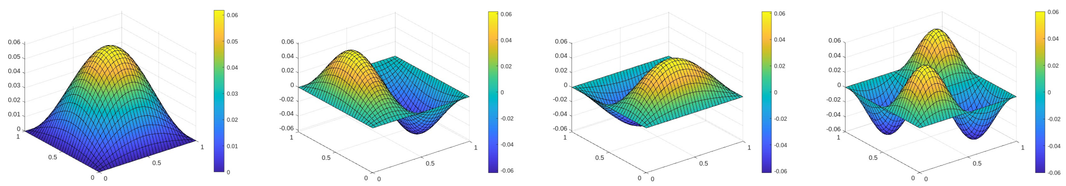

Figure 3: Eigenfunctions $u_{1h}, u_{2h}, u_{3h}, u_{4h}$. Source: Figure created by the author

4.3 Test Circle-Shaped Domain

This test reports the approximation of the first four lowest eigenvalues associated with **VEP-CP** and **VEP-SSP** for a circle-shaped plate defined by and $\Omega_C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/4\}$.

DOI: 10.5540/03.2026.012.01.0241

Tables 4 and 5 illustrate the double order of convergence of the method proposed in this work. In addition, Figure 4 shows the eigenfunctions $u_{1h}, u_{2h}, u_{3h}, u_{4h}$ of a circle-shaped domain for **VEP-CP**.

Table 4: Lowest eigenvalues of a circle-shaped plate for **VEP-CP**.

$\Omega_{\mathrm{C}}^{\mathrm{v}}$	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 1024$	1.6595e+03	7.1424e+03	7.1480e+03	1.9098e+00
$N_P = 4096$	1.6673e+03	7.2098e+03	7.2104e+03	1.9370e+00
$N_P = 16384$	1.6692e+03	7.2266e+03	7.2266e+03	1.9439e+00
rate of conv.	2.0300e+00	2.0100e+00	1.9400e+00	1.9700e+00
λ_{ih}^E	1.6698e+03	7.2321e+03	7.2323e+03	1.9463e+00
$\Omega_{\mathrm{C}}^{\mathrm{t}}$	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 184$	1.6319e+03	6.8691e+03	6.8875e+03	1.7970e+00
$N_P = 712$	1.6597e+03	7.1368e+03	7.1418e+03	1.9052e+00
$N_P = 2826$	1.6673e+03	7.2084e+03	7.2088e+03	1.9359e+00
rate of conv.	1.8800e+00	1.9000e+00	1.9200e+00	1.8100e+00
λ_{ih}^E	1.6701e+03	7.2347e+03	7.2330e+03	1.9483e+00

Table 5: Lowest eigenvalues of a circle-shaped plate for **VEP-SSP**.

$\Omega_{\mathrm{C}}^{\mathrm{t}}$	λ_{1h}	λ_{2h}	λ_{3h}	λ_{4h}
$N_P = 184$	3.1409e+02	2.8547e+03	2.8582e+03	9.8049e+03
$N_P = 712$	3.1546e+02	2.9007e+03	2.9012e+03	1.0090e+00
$N_P = 2826$	3.1581e+02	2.9126e+03	2.9127e+03	1.0170e+00
rate of conv.	1.9500e+00	1.9500e+00	1.9000e+00	1.8200e+00
λ_{ih}^E	3.1594e+02	2.9168e+03	2.9169e+03	1.0202e+00

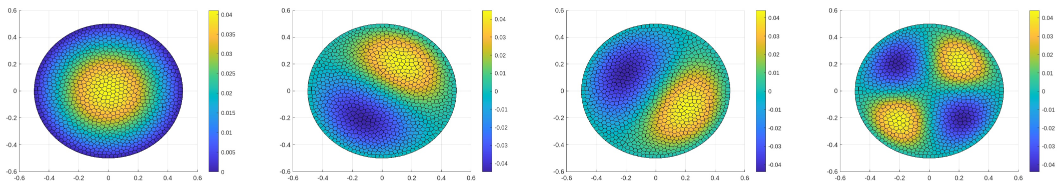

Figure 4: Eigenfunctions $u_{1h}, u_{2h}, u_{3h}, u_{4h}$. Source: Figure created by the author

5 Conclusion

We have considered a virtual element method for solving the vibration problem of a thin elastic plate modeled by the Kirchhoff-Love equations. We derived a weak variational formulation within the Sobolev space H^2 and proposed a discretization scheme using lowest-order non-conforming elements. Our theoretical analysis provides optimal-order error estimates, confirming that the method offers an accurate approximation of both the eigenvalues and the eigenfunctions of the vibration problem. The numerical tests conducted demonstrate the effectiveness of the proposed method, successfully verifying the predicted optimal order of convergence for the eigenvalues. These

DOI: 10.5540/03.2026.012.01.0241

results highlight the potential of the virtual element method as a robust tool for solving vibration problems in thin plates, offering a promising approach for engineering applications where such problems arise.

Acknowledgements

Producto derivado del proyecto INV-CIAS-4167 financiado por la Universidad Militar Nueva Granada-Vigencia 2025. The author is grateful to Vicerrectoría de investigaciones Universidad Militar Nueva Granada for partial financial support (grant INV-CIAS 4167).

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