Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Existence and Uniqueness of Entropy Solutions, and Nonlocal L^1-Stability to the Local Limit and Numerical Applications

Lohan dos Santos[1], Eduardo Abreu[2]
IMECC/Unicamp, Campinas, SP

Abstract. In this work, we present a two-fold result: (1) the generalization of a nonlocal pair interaction model recently introduced in [6] and (2) the construction of a novel semi-discrete approach by using the weak asymptotic method as introduced in [1]. We ensure that the new semi-discrete scheme satisfies an entropy inequality that recovers the local entropy inequality associated to the corresponding nonlocal model, subject to any $C^1(\mathbb{R}^n)$ flux function with initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We will also state and explain the main ideas of some results (this is a work in progress) that allows us to remove the boundedness of the initial data instead of assuming the more restrictive assumption Total Variation Bounded for the proposed new class of semi-discrete convergent schemes. Indeed, by using results of the work [4], we also ensure existence, uniqueness and $L^1(\mathbb{R}^n)$ stability with initial data belonging only to $L^1(\mathbb{R}^n)$. Some numerical results are presented and discussed in the efforts to justify the reliability, but also verifying the theory acquired.
Keywords. Nonlocal Conservation Laws; Weak Asymptotic Method; Semi-discrete Schemes

1 Introduction

In many models developed to describe physical phenomena, the classical framework of partial differential equations may not be sufficient to accurately capture certain effects as fundamental singularities [5]. To achieve a more precise physical description, nonlocal models have been introduced, as illustrated in [2, 5]. In this work, we develop a semi-discrete formulation and perform a weak asymptotic numerical analysis based on [1, 3] for the entropy solution of a nonlocal model. Consider $x \in \mathbb{R}^n$, $t \in [0, +\infty)$ and $g : \mathbb{R}^2 \to \mathbb{R}$ a C^1 function. We define:

$$D_i^h(g, u(x,t)) = \frac{g\left(u(x,t), u(x+he_i,t)\right) - g\left(u(x-he_i,t), u(x,t)\right)}{h}, \tag{1}$$

where e_i is the i vector of the canonical basis of $\mathbb{R}^n$. Consider the following nonlocal PDE:

$$\begin{aligned} &u_t + \sum_{i=1}^{n} [H_i(u)]_{x_i} + \sum_{i=1}^{n} \int_0^{\delta_i} D_i^h(g_i, u(x,t)) w^{\delta_i}(h) dh = F(u), \\ &u(x,0) = u_0(x), \end{aligned} \tag{2}$$

where each $w^{\delta_i}(h)$ is a Dirac sequence with $w^{\delta_i}(0) = 0$ and each H_i is $C^1(\mathbb{R})$ functions. We will always associate each nonlocal therm of (2) with the local associated case $g_i(u,u) = G_i(u)$.

This is a mixed model (local and non-local) that generalizes [6, 7]. In this work, we will state and explain the following results:

[1] v233745@dac.unicamp.br
[2] eabreu@unicamp.br

i) the equation (2) has a unique generalized entropy solution.

ii) When $\delta_i \to 0$, we will have the solution of the model (2) converging to the solution of the associated equation with the local function $[H_i(u) + G_i(u)]_x$.

iii) If g_i, H_i has bounded derivative, so the same statements i) and ii) remain valid even when considering only $u_0 \in L^1(\mathbb{R})$.

2 Existence and Uniqueness of Entropy Solution

We will develop a notion of entropy solutions for generalized balance laws based on [8]. Consider the following Cauchy problem

$$\begin{aligned} u_t + div[H(u)] &= W(u), \\ u(x,0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{aligned} \tag{3}$$

where W is a operator on $L^\infty(\mathbb{R}^n)$ such that, for each $R > 0$ a positive constant, we have

$$\int_0^T \int_{|x|\leq R+C_2t} |W(u) - W(v)| \, dx \, dt \leq \int_0^T \int_{|x|\leq R+C_2t} K(t)|u-v| \, dx \, dt, \tag{4}$$

where $K(t) = K(t, M)$ with $M = \max\{||u||_\infty, ||v||_\infty\}$ and C_2 is some positive constant such that

$$\sup_{|x|,|y|\leq M} \frac{|H(x) - H(y)|}{|x-y|} \leq C_2. \tag{5}$$

We say that u is entropy solution of (3) if it is a weak solution and attain the following inequality

$$\int_A |u-c|\phi_t + sign(u-c)[H(u) - H(c)] \cdot \nabla\phi dA + sign(u-c)W(u)dV \geq 0. \tag{6}$$

With this notion of entropy, we ensure the following result:

Theorem 2.1. *Consider the PDE* (3) *with initial data* $u_0(x) \in L^\infty(\mathbb{R}^n)$. *Suppose that* u, v *are entropy solutions of* (6) *with initial data* u_0, v_0 *respectivelly, so we have the following inequality*

$$\int_{|x|\leq R} |u(x,t) - v(x,t)|dx \leq \int_{|x|\leq R+C_2t} |u(x,0) - v(x,0)|dx e^{\int K(t)dt}. \tag{7}$$

Moreover, if in additional we impose $u_0(x) \in L^1(\mathbb{R}^n)$ *and*

$$\int_0^T \int_{\mathbb{R}^n} |W(u) - W(v)|dxdt \leq \int_0^T \int_{\mathbb{R}^n} K(t)|u-v|dxdt, \tag{8}$$

we have

$$\int_{\mathbb{R}^n} |u(x,t) - v(x,t)|dx \leq \int_{\mathbb{R}^n} |u(x,0) - v(x,0)|dx e^{\int K(t)dt}. \tag{9}$$

For the equation (2), due the hypothesis of $w^{\delta_i}/h \in L^1(0,h)$, we have that the operator

$$W(u) = \sum_{i=1}^{n} \int_0^{\delta_i} D_i^h(g_i, u(x,t)) w^{\delta_i}(h)dh - F(u) \tag{10}$$

DOI: 10.5540/03.2026.012.01.0284

is uniformly Lipschitz on $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, so the equation (2) has a unique entropy solution. On the previous papers [6, 7] about the equation (2), they consider the one dimensional case

$$u_t + \int_0^\delta \frac{g\left(u(x,t), u(x+h,t)\right) - g\left(u(x-h,t), u(x,t)\right)}{h} w^\delta(h)dh = 0 \tag{11}$$

with hypothesis of monotonicity of g. Here we generalize for any $C^1(\mathbb{R}^2)$ flux function $g(\cdot,\cdot)$. This result was achieved using the fact that any $C^1(\mathbb{R}^2)$ flux function can be approximated by a monotonic one. To achieve a entropy solution of (2) and the convergence of the nonlocal mode to the local, we use the following semi discrete scheme

$$\begin{aligned}(u_\varepsilon)_t = &-\sum_{i=1}^{n} D_i^{\varepsilon_i}(h_i, u_\varepsilon) - \sum_{i=1}^{n}\sum_{k=1}^{r_i}[g_i(u_{\varepsilon_i}, u_{\varepsilon_i+k}) - g_i(u_{\varepsilon_i-k}, u_{\varepsilon_i})]W_k^i + F(u_\varepsilon) \\ &-\sum_{i=i}^{n} K_{h_i}(t)\frac{(u_\varepsilon - u_{\varepsilon_{i+1}}) - (u_{\varepsilon_{i-1}} - u_\varepsilon)}{\varepsilon_i} - \sum_{i=1}^{n}\sum_{k=1}^{r_i} K_{g_i}(t)[(u_{\varepsilon_i} - u_{\varepsilon_i+k}) - (u_{\varepsilon_i-k} - u_{\varepsilon_i})]W_k^i,\end{aligned} \tag{12}$$

where $W_k^i = \dfrac{1}{k\varepsilon_i}\int_{(k-1)\varepsilon_i}^{k\varepsilon_i} w^{\delta_i}(h)dh$, $K_{h_i}(t)$ and $K_{g_i}(t)$ are $C^1(\mathbb{R})$ function such that,

$$K_{h_i}(t) \geq \sup_{x,y\in[M(t),M(t)]} (|(h_i)_x(x,y)| + |(h_i)_y(x,y)|), \tag{13}$$

$$K_{g_i}(t) \geq \sup_{x,y\in[M(t),M(t)]} (|(g_i)_x(x,y)| + |(g_i)_y(x,y)|), \tag{14}$$

where $M(t) = ||u_0||_\infty e^{Kt}$ and we also consider the following hypothesis

i) Consistency:

$$h_i(u,u) = H_i(u). \tag{15}$$

ii) $g_i, h_i \in C^1(\mathbb{R}^2)$.

iii) F is a $C^1(\mathbb{R})$ function such that $F(0) = 0$ with Lipschitz constant K on $\mathbb{R}$ i.e.

$$|F(x)| \leq Kx \quad \forall x \in \mathbb{R}. \tag{16}$$

For the homogeneous case ($F = 0$), we have a simple choice of K_{h_i} and K_{g_i} that is

$$K_{h_i} = \sup_{x,y\in[M,M]} (|(h_i)_x(x,y)| + |(h_i)_y(x,y)|), \tag{17}$$

$$K_{g_i} = \sup_{x,y\in[M,M]} (|(g_i)_x(x,y)| + |(g_i)_y(x,y)|), \tag{18}$$

where $M = ||u_0||_\infty$. Under this hypothesis, we ensure that the scheme (12) is convergent and attain the inequality $||u_\varepsilon(t)||_\infty \leq ||u_0||_\infty e^{Kt}$, moreover, if we take $\delta_i \to 0$, the nonlocal therm converges for the local associated $[G(u)]_{x_i}$.

3 Numerical Experiments

In this section, we will numerically solve the inviscid Burgers equation

$$u_t + \left[\frac{u^2}{2}\right]_x = 0, \tag{19}$$

and the Buckley-Leverett equation

$$u_t + \left[\frac{u^2}{u^2 + \frac{(1-u)^2}{2}} \right]_x = 0, \tag{20}$$

and compare with the nonlocal model using the Lagrangian-Eulerian numerical flux

$$F(x,y) = \frac{\Delta x}{\Delta t}(x-y) + \left(\frac{H(x)}{x} + \frac{H(y)}{y} \right)(x+y).$$

We will always plot both the local and nonlocal models for reference, but we will present separately the analysis of the convergence of the model (with δ fixed) and the convergence of the nonlocal model to the local model.

3.1 Convergence of the Nonlocal Numerical Scheme

In this subsection, we verify numerical convergence of the scheme for the nonlocal model.
First, we consider the initial condition

$$u_0(x) = -\sin(\pi x). \tag{21}$$

In the labels, we describe the number of cells and the nonlocal graphics. We are considering the domain $(x,t) \in [-1,1] \times [0,1]$, with $\Delta x = 1/128, 1/256, 1/512$, $\Delta t = \Delta x/4$, and $\delta = 1/32$ at the time $T = 0.125$.

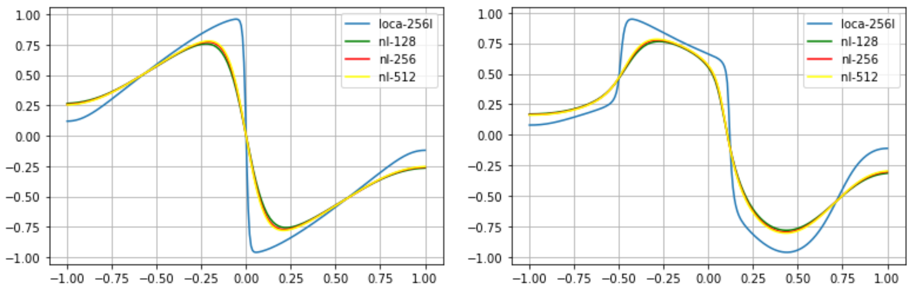

Figure 1: Burger's equation and Buckley Leverett equations $T = 0.125, \delta = 1/32$. Source: Author

For the Riemann problems, we consider the domain $(x,t) \in [-1,1] \times [0,1]$ with $\Delta x = 1/128, 1/256, 1/512, 1/1024$, $\Delta t = \Delta x/4$, and $\delta = 1/32$ at the time $T = 0.125$.
Consider the Burgers equation, Eq. (19), with initial data:

$$u_0(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0 \end{cases} \tag{22}$$

and the Buckley-Leverett equation, Eq. (20), with initial data:

$$u_0(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{23}$$

DOI: 10.5540/03.2026.012.01.0284

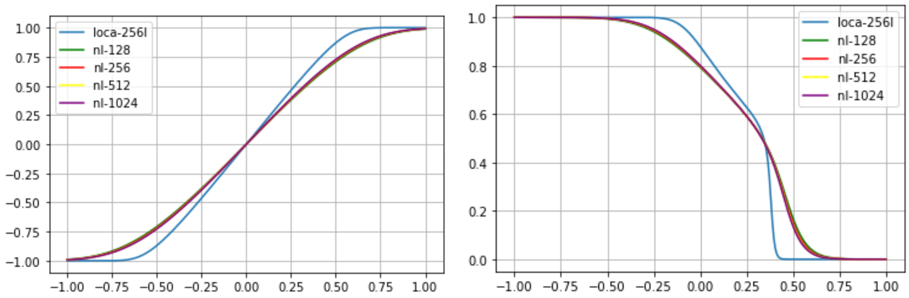

Figure 2: Burger's equation equation and Buckley Leverett $T = 0.125, \delta = 1/32$. Source: Author.

3.2 Convergence for the Local Model

Here we are going to keep $\delta/\Delta x = 1/8$ and refine both at same time. We are considering $(x, t) \in [-1, 1] \times [0, 1]$ with and $\Delta x = 1/128, 1/256, 1/512, 1/1024$, $\Delta t = \Delta x/4$. For the Burgers equation we consider $u_0(x) = -sin(\pi x)$. On the Buckley Leverett equation we consider the initial data (23).

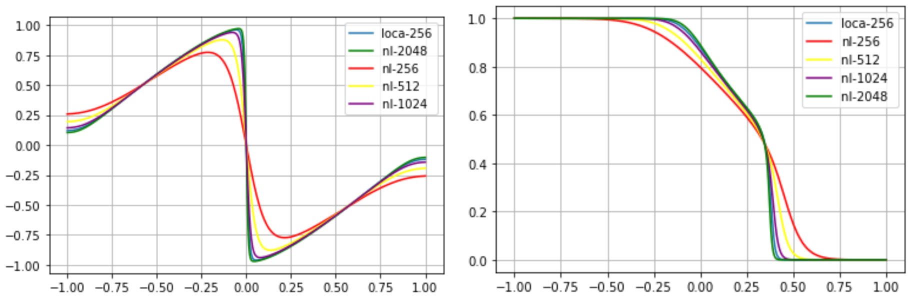

Figure 3: Burger's equation and Buckley Leverett equation $T = 0.125$. Source: Author

4 Unbounded initial Data for DVV Schemes

After to stablish existence and L^1 stability results for the equation (2), inspired on the works of [4, 9], we wish to generalize the hypothesis under the initial data, that is, we consider only $u_0 \in L^1(\mathbb{R}^n)$. The main difficult about a non linear hyperbolic model with unbounded initial data is to ensure local integrability of the non linear term, that is, the volume integral

$$\int_{\mathbb{R}} H(u)\phi_x dx, \tag{24}$$

may not be well defined for all $\phi \in C_c^\infty(\mathbb{R}^n)$. So we consider the hypothesis of uniform Lipschitz of the functions H_i, g_i on (2). This hypothesis holds for some models of fluid dynamic, as the Buckley-leveret equation (20). This led us to define a more general class of numerical schemes. First, we need to introduce some concepts.

We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ has *D-variation bounded* if

$$\lim_{h \to 0} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| \frac{[u(x) - u(x - e_i h)] - [u(x + he_i) - u(x)]}{h} \right| dx < \infty. \tag{25}$$

On the next result, we establish the connection of this class of functions with BV functions.

Lemma 4.1. *Consider $f : \mathbb{R}^n \to \mathbb{R}$ a BV function, then*

$$\frac{1}{n}\lim_{h\to 0}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\left|\frac{u(x)-u(x-e_ih)}{h}\right|dx \leq TV(u). \tag{26}$$

Moreover, the following limit holds

$$\lim_{h\to 0}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\left|\frac{[u(x)-u(x-e_ih)]-[u(x+he_i)-u(x)]}{h}\right|dx = 0. \tag{27}$$

Now introduce a class of methods that generalize TVS. Denote by

$$D_i^2(u) = \frac{[u(x)-u(x-e_ih)]-[u(x+he_i)-u(x)]}{h}. \tag{28}$$

We say that a scheme is DVV(double variation vanishing) if

$$\lim_{h\to 0}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\left|D_i^2(u_\varepsilon(t,x))\right|dx \to 0 \tag{29}$$

uniformly on ε. Due the lemma(4.1), we have that any TVS method is DDV, so we have the continence under the hypothesis of initial data $u_0 \in BV \cap L^\infty$

Monotone ⊂ TVD ⊂ TV Stable ⊂ DVV ⊂ DVC.

For this analysis, we use the following semi-discrete scheme

$$(u_\varepsilon)_t = -\sum_{i=1}^{n} D_i^{\varepsilon_i}(h_i,u_\varepsilon) - \sum_{i=1}^{n}\sum_{k=1}^{r_i}[g_i(u_{\varepsilon_i},u_{\varepsilon_i+k}) - g_i(u_{\varepsilon_i-k},u_{\varepsilon_i})]W_k^i + F(u_\varepsilon) \tag{30}$$

$$u_\varepsilon(x,0) = u_0 \in L^1(\mathbb{R}^n). \tag{31}$$

and we ensure the following theorem

Theorem 4.1. *Consider the following PDE* (30) *with $g_i, h_i \in C^1(\mathbb{R}^n)$ are global Lipschitz functions with Lipschitz constant K, $g_i(0,0) = h_i(0,0) = 0$, attain* (15) *and F attain* (16)*, so the PDE has a unique local classical solution on $t \in [0,+\infty)$.*

The following result ensure L^1 stability of our approximation

Theorem 4.2. *Consider $u_\varepsilon, v_\varepsilon$ solutions of* (30) *with initial data u_0, v_0 respectively, so:*

$$||u_\varepsilon(t_1)-v_\varepsilon(t_1)||_{L^1(\mathbb{R}^n)}e^{-2Kt_1} + \int_0^T ||W(u_\varepsilon(t)) - W(v_\varepsilon(t))||_{L^1(\mathbb{R}^n)}e^{-2kt}dt \geq ||u_\varepsilon(t_2)-v_\varepsilon(t_2)||_{L^1(\mathbb{R}^n)}e^{-2Kt_2} \tag{32}$$

where $t_2, t_1 \in [0,T]$, $t_1 < t_2$ and

$$W(u_\varepsilon) = K\left(\sum_{i=i}^{n}\frac{(u_\varepsilon - u_{\varepsilon_{i+1})} - (u_{\varepsilon_{i-1}} - u_\varepsilon)}{\varepsilon_i} + \sum_{i=1}^{n}\sum_{k=1}^{r_i}[(u_{\varepsilon_i} - u_{\varepsilon_i+k}) - (u_{\varepsilon_i-k} - u_{\varepsilon_i})]W_k^i\right). \tag{33}$$

With the results above we ensure that if the scheme (30) defines a DVV method, so it converges for the entropy solution of (2) with $u_0 \in L^1(\mathbb{R}^n)$, so we have the following result

Theorem 4.3. *Consider the PDE* (2) *with $H_i, F \in C^1(\mathbb{R}), g_i \in C^1(\mathbb{R}^2)$ with a uniform Lipschitz constant K and $u_0(x) \in L^1(\mathbb{R}^n)$. So it has a unique entropy solution, moreover, when $\delta_i \to 0$, we will have the solution of the model* (2) *converging to the solution of the associated equation with the local function $[H_i(u) + G_i(u)]_x$.*

5 Conclusions

In this work, we presented and discussed a generalization of a nonlocal pair iteration model, firstly introduced on [6] and a new semi-discrete approach via the weak asymptotic method [1]. Due to our semi-discrete approach, we were able to develop a class of robust numerical methods, in a solid basis, that capture the correct nonlocal solution and recover the nonlocal limit to the local form of the underlying model. Moreover, we found the effectiveness of the combination of the weak asymptotic method and Kruzhkov's theory of balance laws to ensure the existence, uniqueness and L^1 -stability of the unique entropy solution for the nonlocal nonlinear equation (1). The new approach introduced in this work also allows us to remove the hypothesis of boundness of the initial data and establish uniqueness of weak solutions broader than one provided by classical numerical methods in the specific TV-class of regularity. Finally, numerical experiments were presented to justify the reliability of the proposed semi-discrete scheme, which is also supported with respect to the acquired theory for nonlocal models of type (1).

References

[1] E. Abreu, M. Colombeau, and E. Panov. "Weak asymptotic methods for scalar equations and systems". In: **Journal of mathematical analysis and applications** 444.2 (2016), pp. 1203–1232. DOI: 10.1016/j.jmaa.2016.06.047.

[2] E. Abreu, W. Juajibioy J.and Lambert, et al. "Semi-discrete Lagrangian–Eulerian approach based on the weak asymptotic method for nonlocal conservation laws in several dimensions". In: **Journal of Computational and Applied Mathematics** 458 (2025), p. 116325. DOI: 10.1016/j.cam.2024.116325.

[3] E. Abreu and J. Pérez. "A fast, robust and simple Lagrangian–Eulerian solver for balance laws and applications". In: **Computers & Mathematics with Applications** 77.9 (2019), pp. 2310–2336. DOI: 10.1016/j.camwa.2018.12.019.

[4] M. G. Crandall. "The semigroup approach to first order quasilinear equations in several space variables". In: **Israel Journal of Mathematics** 12 (1972), pp. 108–132. DOI: 10.1007/BF02764657.

[5] M. D'Elia, Q. Du, C. Glusa, M. Gunzburger, X. Tian, and Z. Zhou. "Numerical methods for nonlocal and fractional models". In: **Acta Numerica** 29 (2020), pp. 1–124. DOI: 10.1017/S096249292000001X.

[6] Q. Du, Z. Huang, and P. G. LeFloch. "Nonlocal conservation laws. A new class of monotonicity-preserving models". In: **SIAM Journal on Numerical Analysis** 55.5 (2017), pp. 2465–2489. DOI: 10.1137/16M1105372.

[7] U. S. Fjordholm and A. M. Ruf. "Second-order accurate TVD numerical methods for nonlocal nonlinear conservation laws". In: **SIAM Journal on Numerical Analysis** 59.3 (2021), pp. 1167–1194. DOI: 10.1137/20M1360979.

[8] H. Frid. "Uniqueness of solutions to hyperbolic balance laws in several space dimensions". In: **Communications in partial differential equations** 14.8-9 (1989), pp. 959–979. DOI: 10.1080/03605308908820638.

[9] D. Serre and L. Silvestre. "Multi-dimensional Burgers equation with unbounded initial data: well-posedness and dispersive estimates". In: **Archive for Rational Mechanics and Analysis** 234 (2019), pp. 1391–1411. DOI: 10.1007/s00205-019-01414-4.

