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The Stabilized Three-Field Domain Decomposition Method

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Abstract. In this work we consider the Three-Field finite element formulation for elliptic problems. In the original setting, not all combinations of spaces will lead to stable formulations, and two independent inf-sup conditions must be satisfied. One way to relax such constraint is to introduce stabilizations terms, allowing more space choices. Our goal is to propose a stabilized scheme that allows different combinations of polynomials for the unknowns involved. We explore here the possibility of employing the three-field formulation as a direct method, i.e., no submeshes involved. We show coercivity and convergence results in a suitable mesh dependent norm. Efficient implementation of the method is still possible, as static condensation of the unknowns can be performed at the element level, in parallel. We present numerical results displaying the performance of the method.

Keywords. Hybrid Method, Stabilized Method, Hybrid Mixed Method, Finite Element Method, Multiscale-Hybrid-Hybrid Mixed Method, Numerical Nethod

1 Introduction

We introduce some definitions and notations commonly adopted to construct weak formulations in broken function spaces associate with Discontinuous Galerkin (DG) and hybrid methods. Let the following second order elliptic model problem: Find the scalar variable $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is an open bounded domain with a Lipschitz continuous boundary Γ, Δ is the Laplacian operator and $f \in L^2(\Omega)$ is the source term . We recall some basic results on well known classes of primal DG and hybrid formulations for elliptic problems. Let $\mathcal{T}_h = \{K\}$ be a regular finite element partition of the domain Ω and let $\mathcal{E}_h := \{e; e \text{ is an edge of K for all } K \in \mathcal{T}_h\}$ denote the set of all edges of the mesh $\mathcal{T}_h$, $\mathcal{E}_h^\partial := \mathcal{E}_h \cap \Gamma$ is the set of edges of $\mathcal{E}_h$ that lie on the boundary Γ and $\mathcal{E}_h^0 := \mathcal{E}_h \setminus \mathcal{E}_h^\partial$ the set of interior edges. We set $\mathbf{n}$ the unit normal vector on Γ and $\mathbf{n}_K$ the exterior normal vector defined on ∂K, $K \in \mathcal{T}_h$. Consider also the following broken Sobolev spaces

$$H^1(\mathcal{T}_h) := \{v \in \mathrm{L}^2(\Omega); v|_K \in \mathrm{H}^1(K), K \in \mathcal{T}_h\} \qquad \Lambda := \prod_{K \in \mathcal{T}_h} \mathrm{H}^{-1/2}(\partial K) \tag{2}$$
$$\mathrm{H}_0^{1/2}(\mathcal{E}_h) := \{v|_{\mathcal{E}_h},\ v \in \mathrm{H}_0^1(\Omega)\}$$

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The three-field domain decomposition method [5] associated to the model problem (1) is: Find $[u, \lambda, \hat{u}] \in \mathrm{H}^1(\mathcal{T}_h) \times \Lambda \times \mathrm{H}_0^{1/2}(\mathcal{E}_h)$ such that

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda (v - \hat{v}) \, ds - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu (u - \hat{u}) \, ds = \sum_{K \in \mathcal{T}_h} \int_K f v \, dx \tag{3}$$

for all $[v, \mu, \hat{v}] \in \mathrm{H}^1(\mathcal{T}_h) \times \Lambda \times \mathrm{H}_0^{1/2}(\mathcal{E}_h)$.

2 Hybrid Formulation

We start by introducing the concept of jump $[\![\cdot]\!]$ and average $\{\cdot\}$ of functions on the interfaces $e \in \mathcal{E}_h$. Let $e \in \mathcal{E}_h^0$ be an edge shared by elements $K_1, K_2 \in \mathcal{T}_h$. Consider $\mathbf{n}_e$ the unit normal vector on e pointing from K_1 to K_2. For a function w defined in $K_1 \cup K_2$, possibly discontinuous across e, we denote $w_i := (w|_{K_i})|_e$, $i = 1, 2$. If $v \in L^2(K_1 \cup K_2)$ is a scalar function and $\tau \in [L^2(K_1 \cup K_2)]^d$ is a vectorial function, we then define its averages and jumps as

$$\begin{aligned} \{v\} &:= \frac{1}{2}(v_1 + v_2) & \qquad [\![v]\!] &:= (v_1 - v_2)\mathbf{n}_e \\ \{\tau\} &:= \frac{1}{2}(\tau_1 + \tau_2) & \qquad [\![\tau]\!] &:= (\tau_1 - \tau_2) \cdot \mathbf{n}_e \end{aligned} \tag{4}$$

For a boundary edge $e \in \mathcal{E}_h^\partial \cap \partial K_1$, we consider simply the restrictions

$$[\![v]\!] = \{v\} := v_1 \qquad\qquad [\![\tau]\!] = \{\tau\} = \tau_1. \tag{5}$$

We refer [6] for a more precise definition. Following [1], its possible to verify the identity

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \cdot \mathbf{n}_K v \, ds = \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\tau\} \, ds + \int_{\mathcal{E}_h^0} \{v\} [\![\tau]\!] \, ds. \tag{6}$$

We approximate the infinite dimensional spaces in (2) by the following broken polynomial spaces

$$\begin{aligned} &V_h^k := \{v_h \in L^2(\Omega);\ v_h|_K \in \mathbb{P}_k(K),\ \forall\, K \in \mathcal{T}_h\} \subset H^1(\mathcal{T}_h) \\ &Q_h^n = \{\lambda_h \in L^2(\partial K);\ \lambda_h|_e = \mathbb{P}_n(e),\ \forall\, e \in \partial K,\ \forall\, K \in \mathcal{T}_h\} \subset \Lambda \\ &M_h^l = \{\hat{v}_h \in \mathcal{C}^0(\mathcal{E}_h);\ \hat{v}_h|_e \in \mathbb{P}_l(e),\ \forall\, e \in \mathcal{E}_h^0,\ \hat{v}|_e = 0,\ \forall\, e \in \mathcal{E}_h^\partial\} \subset H_0^{1/2}(\mathcal{E}_h). \end{aligned} \tag{7}$$

In what follows we consider the notation: Let $W_h := V_h^k \times Q_h^n \times M_h^l$ and $\underline{u}_h, \underline{v}_h \in W_h$ such that $\underline{u}_h := (u_h, \lambda_h, \hat{u}_h)$ and $\underline{v}_h := (v_h, \mu_h, \hat{v}_h)$ for $u_h, v_h \in V_h^k$, $\lambda_h, \mu_h \in Q_h^k$ and $\hat{u}_h, \hat{v}_h \in M_h^l$. Given $\underline{u}_h = (u_h, \lambda_h, \hat{u}_h) \in W_h$, we write $\underline{u}_h^* := (u_h, -\lambda_h, \hat{u}_h) \in W_h$.

2.1 The Multiscale-Hybrid-Hybrid-Mixed Method - MH^{2}M

The MH^{2}M method is a numerical method to solve a symmetric positive definite global problem posed on the interfaces of the mesh. This method comes from the three-field formulation (3) going through static condensation steps, under suitable compatibility conditions. This method can be presented as: Find $\underline{u}_h \in W_h$ such that

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda_h (v_h - \hat{v}_h) \, ds - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu_h (u_h - \hat{u}_h) \, ds = \sum_{K \in \mathcal{T}_h} \int_K f v_h \, dx, \tag{8}$$

for all $\underline{v}_h \in W_h$. Stability of the MH^{2}M formulation depends on the choice of the finite element spaces V_h^k, Q_h^n and M_h^l. More precisely, the the following two inf-sup conditions [4] must be satisfied

$$\sup_{v_h \in V_h^k} \frac{\int_{\partial K} \mu_h v_h \, ds}{\|v_h\|_{h,K}} \geq C\|\mu_h\|_{\partial K} \qquad \sup_{\mu_h \in Q_h^n} \frac{\int_{\partial K} \mu_h \hat{v}_h \, ds}{\|\mu_h\|_{\partial}} \geq C\|\hat{v}_h\|_{\partial K} \tag{9}$$

An unconditionally stable version of the Three-field formulation is presented next.

3 The Stabilized Three-Field Domain Decomposition Method

By adding two stabilization terms [4] in the three-field formulation (8), the full stabilized version of this method is proposed as: Find $\underline{u}_h \in W_h$ such that

$$A_H(\underline{u}_h, \underline{v}_h) = F(\underline{v}_h), \tag{10}$$

for all $\underline{v}_h \in W_h$, where the bilinear form $A_H : W_h \times W_h \to \mathbb{R}$ is defined as

$$\begin{aligned} A_H(\underline{u}_h, \underline{v}_h) := &\sum_{K\in\mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h \, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \lambda_h (v_h - \hat{v}_h) \, ds \\ &- \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h (u_h - \hat{u}_h) \, ds + \alpha \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla u_h \cdot n_K - \lambda_h)(\nabla v_h \cdot n_K - \mu_h) \, ds \\ &+ \beta \sum_{K\in\mathcal{T}_h} \int_{\partial K} (u_h - \hat{u}_h)(v_h - \hat{v}_h) \, ds \end{aligned} \tag{11}$$

with $\beta = \beta_0/h$, $\beta_0 > 0$ e $\alpha = -\alpha_0 h$, $\alpha_0 > 0$. From the identity (6), the bilinear form (11) can be written as

$$\begin{aligned} A_H(\underline{u}_h, \underline{v}_h) := &\sum_{K\in\mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h \, dx - \sum_{e\in\mathcal{E}_h^0} \int_e (\{\mu_h\}[\![u_h]\!] + \{\lambda_h\}[\![v_h]\!] \, ds \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \frac{\beta}{2} [\![u_h]\!] \cdot [\![v_h]\!] \, ds - \sum_{e\in\mathcal{E}_h^0} \int_e ([\![\lambda_h]\!](\{v_h\} - \hat{v}_h) + [\![\mu_h]\!](\{u_h\} - \hat{u}_h)) \, ds \\ &+ \alpha \sum_{e\in\mathcal{E}_h} \int_e (\nabla u_h \cdot n_e - \lambda_h)(\nabla v_h \cdot n_e - \mu_h) \, ds \\ &+ \sum_{e\in\mathcal{E}_h^0} \int_e 2\beta(\hat{u}_h - \{u_h\})(\hat{v}_h - \{v_h\}) \, ds \end{aligned} \tag{12}$$

for all $\underline{v}_h \in W_h$.

Remark 3.1. *Note that our stabilization scheme differs from the one proposed in [3] since it precludes the regularity of the operator inside each element. That is particularly useful when the problem is determined by L^∞ coefficients, or has multiscale aspects.*

3.1 Stability of the Method

In this section we show the stability of the method for any choice of the polynomials degree of the broken spaces V_h^k, Q_h^n and M_h^l. The stability is verified for two cases: the first one is the full stabilization, where both Lagrange multipliers λ_h and $\hat{u}_h$ are stabilized; the second one, only the local Lagrange multiplier λ_h is stabilized. We prove the coercivity of the bilinear form $A_H(\cdot,\cdot)$, independent of the choice of finite element spaces, in the following norm

$$\|\underline{u}_h\|_{HH}^2 := \sum_{K\in\mathcal{T}_h}\int_K |\nabla u_h|^2\,dx + h\sum_{K\in\mathcal{T}_h}\int_{\partial K}\lambda_h^2\,ds + \frac{1}{h}\sum_{e\in\mathcal{E}_h^0}\int_e |[\![u_h]\!]|^2\,ds + \frac{1}{h}\sum_{e\in\mathcal{E}_h^\partial}\int_e |u_h|^2\,ds + \frac{1}{h}\sum_{e\in\mathcal{E}_h^0}\int_e |\hat{u}_h - \{u_h\}|^2\,ds \tag{13}$$

Theorem 3.1. *(Full stabilization) Let $\alpha_0 > 0$ sufficiently small and $\beta_0 > 0$. There exists a constant $\gamma > 0$, independent of h, such that*

$$A_H(\underline{u}_h, \underline{u}_h^*) \geq \gamma\|\underline{u}_h\|_{HH}^2 \tag{14}$$

for all $\underline{u}_h \in W_h$.

Theorem 3.2. *(Local stabilization and unconditional stability) Let $\beta_0 = 0$ and $\alpha = -\alpha_0 h$ in the bilinear form $A_H : W_h \times W_h \to \mathbb{R}$ introduced in (11). Define $\overline{\mu}_h \in Q_h^n$, as*

1. *for $n \geq l$, $\overline{\mu}_h := -\lambda_h + \frac{1}{\alpha}(u_h - \hat{u}_h)$,*
2. *for $n < l$, $\overline{\mu}_h := -\lambda_h + \frac{1}{\alpha}\Pi_h^n(u_h - \hat{u}_h)$,*

where, for an edge $e \in \mathcal{E}_h$, the local $L^2(e)$-projection $\Pi_h^n v_h$ of $v_h \in V_h^k$ is given by

$$\int_e \Pi_h^n v_h \mu_h\,ds = \int_e v_h\mu_h\,ds, \quad \forall\,\mu_h \in \mathbb{P}_n(e). \tag{15}$$

Then, there exists a constant $\zeta > 0$ independent of h such that

$$A_H(\underline{u}_h, \underline{u}_h^*) \geq \zeta\|\underline{u}_h\|_{HH}^2. \tag{16}$$

for all $\underline{u}_h \in W_h$, and such that $\underline{u}_h^* := [u_h, \overline{\mu}_h, \hat{u}_h]$.

3.2 Static Condensation

For the stabilized method (10), the local variables u_h, defined in each element $K \in \mathcal{T}_h$, and λ_h defined on the boundary ∂K, can be eliminated at the element level by solving the following set of problems in the pair $[u_h, \lambda_h]$:

Local problems: For each $K \in \mathcal{T}_h$, find $[u_h, \lambda_h] \in V_h^k \times Q_h^n$ such that

$$A_H^K([u_h,\lambda_h],[v_h,\mu_h]) + B_H^K(\hat{u}_h,[v_h,\mu_h]) = F([v_h,\mu_h]), \quad \forall\,[v_h,\mu_h] \in V_h^k \times Q_h^n \tag{17}$$

where the bilinear form $A_H^K : V_h^k \times Q_h^n \to \mathbb{R}$ is such that

$$A_H^K([u_h,\lambda_h],[v_h,\mu_h]) := \int_K \nabla u_h\cdot\nabla v_h\,dx - \int_{\partial K}\lambda_h v_h\,ds - \int_{\partial K}\mu_h u_h\,ds + \int_{\partial K}\alpha(\nabla u_h\cdot\mathbf{n}_K - \lambda_h)(\nabla v_h\cdot\mathbf{n}_K - \mu_h)\,ds + \int_{\partial K}\beta u_h v_h\,ds, \tag{18}$$

The forms $B_H^K : M_h^l \times V_h^k \times Q_h^n \to \mathbb{R}$ and $F : V_h^k \times Q_h^n \to \mathbb{R}$ are defined as

$$B_H^K(\hat{u}_h,[v_h,\mu_h]) := \int_{\partial K}\mu_h\hat{u}_h\,ds + \int_{\partial K}\beta\hat{u}_h v_h\,ds, \tag{19}$$

$$F([v_h,\hat{v}_h]) := \sum_{K\in\mathcal{T}_h}\int_K f_h v_h\,dx, \tag{20}$$

for all $[v_h,\mu_h,\hat{v}_h] \in V_h^k \times Q_h^n \times M_h^l$.

Global problem: Find $\hat{u}_h \in M_h^l$ such that

$$\sum_{K\in\mathcal{T}_h} B_H^K(\hat{v}_h, [u_h, \lambda_h]) + \sum_{K\in\mathcal{T}_h} C_H^K(\hat{u}_h, \hat{v}_h) = 0, \quad \forall\, \hat{v}_h \in M_h^l \tag{21}$$

with $B_H^K(\cdot,\cdot)$ as in (19), and $C_H^K \in \mathcal{L}_2(M_h^l;\mathbb{R})$ is such that

$$C_H^K(\hat{u}_h, \hat{v}_h) := \int_{\partial K} \beta \hat{u}_h \hat{v}_h \, ds \tag{22}$$

for all $\hat{v}_h \in M_h^l$. The stability of the local bilinear form $A_H^K(\cdot,\cdot)$ follows from

$$\begin{aligned} A_H^K([u_h,\lambda_h],[u_h,-\lambda_h]) = &\int_K |\nabla u_h|^2\, dx \\ &+ \alpha \int_{\partial K} |\nabla u_h \cdot \mathbf{n}_K|^2\, ds - \alpha \int_{\partial K} |\lambda_h|^2\, ds + \beta \int_{\partial K} |u_h|^2\, ds \end{aligned} \tag{23}$$

with $\alpha = -\alpha_0 h$ and $\beta = \beta_0/h$, for α_0 sufficiently small and $\beta_0 > 0$.

For $k = n = l$ or $n \geq l$, we can solve exactly the local equation of the multiplier λ_h on each edge $e \in \partial K$, obtaining

$$\lambda_h = \nabla u_h \cdot \mathbf{n}_K + \frac{1}{\alpha}(u_h - \hat{u}_h). \tag{24}$$

Replacing the above expression for λ_h in the stabilized formulation (10) with $\beta = 0$, yields the following weak form

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h\cdot\nabla v_h\, dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\nabla u_h\cdot\mathbf{n}_K)(v_h-\hat{v}_h)\, ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\nabla v_h\cdot\mathbf{n}_K)(u_h-\hat{u}_h)\, ds \\ &\qquad - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{1}{\alpha}(u_h-\hat{u}_h)(v_h-\hat{v}_h)\, ds = \sum_{K\in\mathcal{T}_h}\int_K f v_h\, dx \end{aligned} \tag{25}$$

which is identical to the LDGC weak form [2] with $\alpha = -1/\beta$.

Static condensation of all degree of freedom of $[u_h, \lambda_h]$ is always possible for any stable approximation of MH^{2}M.

3.3 Consistency and Continuity

The following assumptions are needed for consistency and continuity of the stabilized method

Definition 3.1. *Let $W_* := H^2(\mathcal{T}_h) \times L^2(\mathcal{E}_h) \times L^2(\mathcal{E}_h)$ so that $V_h^k \subset H^2(\mathcal{T}_h)$, $Q_h^l \subset L^2(\mathcal{E}_h)$ and $M_h^l \subset L^2(\mathcal{E}_h)$, where $H^2(\mathcal{T}_h) := \{v \in L^2(\Omega),\, v|_K \in H^2(K),\, \forall\, K \in \mathcal{T}_h\}$. Consider henceforth that the bilinear form $A_H(\cdot,\cdot)$ admits an extension in its first argument to the space W_*. We assume that $\lambda \in H^{-1/2}(\partial\Omega) \cap L^2(\mathcal{E}_h)$.*

Let u be the solution of the model problem (1). We get consistency of the stabilized three-field formulation if u, $\lambda := \nabla u \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega)$ and $\hat{u} := \gamma_0(u) \in H_0^{1/2}(\partial\Omega)$ solve (10). So, we can show the following result.

Proposition 3.1. *(Consistency) If u solves the model problem (1), then u, $\lambda := \nabla u \cdot \boldsymbol{n}$ and $\hat{u} := u|_{\mathcal{E}_h}$ solve the stabilized hybrid formulation (10).*

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The boundedness of the bilinear form in the norm $\|\cdot\|_{HH}$ is also verified.

Proposition 3.2. *(Continuity) There exists a constant $\eta > 0$, independent of h, such that*

$$|A_H(\underline{u}_h, \underline{v}_h)| \leq \eta \|\underline{u}_h\|_{HH} \|\underline{v}_h\|_{HH} \tag{26}$$

for all $\underline{u}_h \in W_*$ and all $\underline{v}_h \in W_h$.

3.4 Error Estimate

The next result establish the rate of convergence of the numerical finite element solution $(u_h, \lambda_h \hat{u}_h) \in W_h$ of the proposed method (10) to the exact solution $(u, \lambda, \hat{u})$ of the three-field variational formulation (3).

Theorem 3.3. *Let $\underline{u} := (u, \lambda, \hat{u}) \in W_*$ the weak solution of the model problem* (1)*, with $\lambda := \nabla u \cdot \boldsymbol{n}$ and $\hat{u} := u|_{\mathcal{E}_h}$. Let $\underline{u}_h := (u_h, \lambda_h, \hat{u}_h) \in W_h$ the discrete solution of the stabilized hybrid formulation* (10)*. Then there exist a constant C independent of h such that*

$$\|\underline{u} - \underline{u}_h\|_{HH} \leq C(h^k |u|_{k+1,\Omega} + h^{n+1} |\lambda|_{n+1,\mathcal{E}_h} + h^l |u|_{l+1,\Omega}). \tag{27}$$

4 Numerical Results

Consider the domain $\Omega :=]0:1[\times]0:1[$. Let the second order elliptical problem:

$$\begin{aligned} -\Delta u &= 2(y^2 - y) + 2(x^2 - x), \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{28}$$

The analytical solution is given by $u(x, y) = (x^2 - x)(y^2 - y)$ and is depicted in the Figure 1.

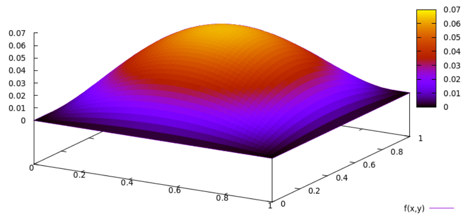

Figure 1: Exact solution. Source: Own elaboration.

The mesh $\mathcal{T}_h$ is a uniform triangulation of Ω. We perform the method with polynomials of degree up to three in the interior of each element $K \in \mathcal{T}_h$ and up to four on the interfaces $e \in \mathcal{E}_h$. The stabilization parameters α and β are determined by solving local generalized eigenvalue problems, coming from localized discrete inverse estimates. We plot in the Figure 2 the curves of error in the norm $\|\cdot\|_{\mathrm{HH}}$ defined in (13) with polynomial interpolation spaces $\mathbb{P}_k - \mathbb{P}_n - \mathbb{P}_l$ to approximate the function, fluz and the trace, respectively. Note that, for all the conbinations (k, n, l) in the Figure 2, the finite element spaces do not satisfies the $\inf - \sup$ conditions (9). We plot the graph of a numerical solution in the Figure 3.

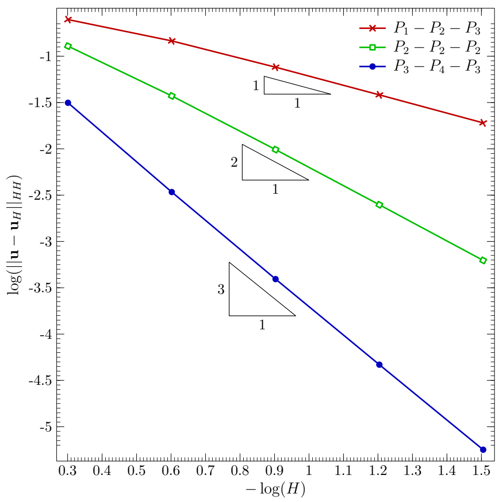

Figure 2: Curves of convergence in the HH-norm. Source: Own elaboration.

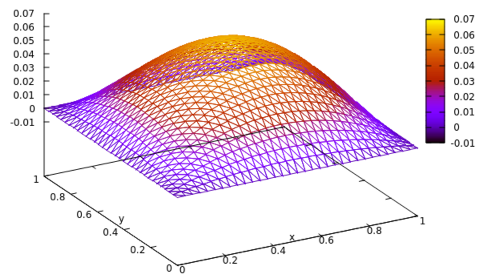

Figure 3: Numerical solution with $h = \sqrt{2}/32$ and $(k, n, l) = (2, 1, 2)$. Source: Own elaboration.

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