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Order preserving and order reversing operators on the class of convex functions in Banach spaces

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Abstract

A remarkable recent result by S. Artstein-Avidan and V. Milman states that, up to pre-composition with affine operators, addition of affine functionals, and multiplication by positive scalars, the only *fully order preserving* mapping acting on the class of lower semicontinuous proper convex functions defined on $\mathbb{R}^n$ is the identity operator, and the only *fully order reversing* one acting on the same set is the Fenchel conjugation. Here *fully order preserving (reversing) mappings* are understood to be those which preserve (reverse) the pointwise order among convex functions, are invertible, and such that their inverses also preserve (reverse) such order. In this paper we establish a suitable extension of these results to order preserving and order reversing operators acting on the class of lower semicontinous proper convex functions defined on arbitrary infinite dimensional Banach spaces.

Key words: order preserving operators, order reversing operators, Fenchel conjugation, convex functions, lower semicontinuous functions, Banach space, involution.

Mathematical Subject Classification (2010): 46N10, 46B10.

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1 Introduction

In their recent paper [4], S. Artstein-Avidan and V. Milman established a remarkable result on certain mappings acting on the class of convex functions defined on $\mathbb{R}^n$. More precisely, let $\mathscr{C}(\mathbb{R}^n)$ be the set of lower semicontinuous proper convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. For $f, g \in \mathscr{C}(\mathbb{R}^n)$, we write $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$. An operator $T : \mathscr{C}(\mathbb{R}^n) \to \mathscr{C}(\mathbb{R}^n)$ is said to be *order preserving* if $T(f) \leq T(g)$ whenever $f \leq g$, and *order reversing* if $T(f) \geq T(g)$ whenever $f \leq g$. We will say that T is fully order preserving (reversing) if, in addition, T is invertible and its inverse T^{-1} is also order preserving (reversing). The main results in [4] state that, up to pre-composition with affine operators, addition of affine functionals, and multiplication by positive scalars, the only fully order preserving mapping acting on $\mathscr{C}(\mathbb{R}^n)$ is the identity operator, and the only fully order reversing one acting on the same set is the Fenchel conjugation (also called Legendre transform, or convex conjugation), meaning the operator T given by $T(f) = f^*$ for all $f \in \mathscr{C}(\mathbb{R}^n)$, where f^* denotes the Fenchel conjugate of f, defined as $f^*(u) = \sup_{x \in \mathbb{R}^n} \{\langle u, x \rangle - f(x)\}$ for all $u \in \mathbb{R}^n$.

We mention that order preserving isomorphisms acting on sets with specific structures had been studied for some time, in some cases in connection with applications to Physics (see. e.g., [13], [14], [15], [16]), but Artstein-Avidan and Milman's work [4] was the first to deal with this issue in connection with the space of convex functions.

The results from [4] were the starting point of several interesting developments, consisting mainly of replacing $\mathscr{C}(X)$ by other sets. Along this line, we mention for instance the characterizations of fully order reversing mappings acting on:

i) the class of s-concave functions (see [3]),

ii) the family of convex functions vanishing at 0 (see [6]),

iii) closed and convex cones, with the order given by the set inclusion, and the polarity mapping playing the role of the Fenchel conjugation (see [17]),

iv) closed and convex sets (see [18]),

v) ellipsoids (see [8]).

Additional recent related results can be found also in [1], [5], [7] and [12].

All these papers deal exclusively with the finite dimensional case. In [20], it is proved that the composition of two order reversing bijections acting on the class of convex functions defined on certain locally convex topological vector spaces is the identity if and only in the same happens with their restrictions to the subset of affine functions, but no characterization of the kind established in [4] is given. To our knowledge, the extension of the above described results in [4] to infinite dimensional spaces remains as an open issue. Adressing it is the main goal of this paper. More precisely, we will generalize the analysis of order preserving and order reversing operators developed by Milman and Artstein-Avidan to the setting of arbitrary Banach spaces.

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2 The order preserving case

The proofs of all the results in this paper can be found in [11].

We start by introducing some quite standard notation. Let X be a real Banach space of dimension at least 2. We denote by X^* and X^{**} its topological dual and bidual, respectively. We define as $\mathscr{C}(X)$ the set of lower semicontinuous proper convex functions $f : X \to \mathbb{R} \cup \{+\infty\}$, where the lower semicontinuity is understood to hold with respect to the strong (or norm) topology in X. For $f \in \mathscr{C}(X)$, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ will denote its Fenchel conjugate, defined as $f^*(u) = \sup_{x \in X}\{\langle u, x\rangle - f(x)\}$ for all $u \in X^*$, where $\langle \cdot, \cdot \rangle$ stands for the duality coupling, i.e. $\langle u, x\rangle = u(x)$ for all $(u, x) \in X^* \times X$. As usual, $f^{**} : X^{**} \to \mathbb{R} \cup \{+\infty\}$ will indicate the biconjugate of f, i.e. $f^{**} = (f^*)^*$. Given a linear and continuous operator $C : X^* \to X^*$, $C^* : X^{**} \to X^{**}$ will denote the adjoint of C, defined by the equation $\langle C^*a, u\rangle = \langle a, Cu\rangle$ for all $(a, u) \in X^{**} \times X^*$. We consider the pointwise order in $\mathscr{C}(X)$, i.e., given $f, g \in \mathscr{C}(X)$, we write $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in X$. We denote as $\mathbb{R}_{++}$ the set of strictly positive real numbers.

The next definition introduces the family of operators on $\mathscr{C}(X)$ whose characterization is the main goal of this paper.

Definition 1. *i) An operator $T : \mathscr{C}(X) \to \mathscr{C}(X)$ is order preserving whenever $f \leq g$ implies $T(f) \leq T(g)$.*

ii) An operator $T : \mathscr{C}(X) \to \mathscr{C}(X)$ is fully order preserving whenever $f \leq g$ iff $T(f) \leq T(g)$, and additionally T is onto.

iii) We define as $\mathscr{B}$ the family of fully order preserving operators $T : \mathscr{C}(X) \to \mathscr{C}(X)$.

We will prove that T belongs to $\mathscr{B}$ if and only if there exist $c \in X$, $w \in X^*$, $\beta \in \mathbb{R}$, $\tau \in \mathbb{R}_{++}$ and a continuous automorphism E of X such that

$$T(f)(x) = \tau f(Ex + c) + \langle w, x\rangle + \beta,$$

for all $f \in \mathscr{C}(X)$ and all $x \in X$.

We begin with an elementary property of the operators in $\mathscr{B}$.

Proposition 1. *If T belongs to $\mathscr{B}$ then,*

i) T is one-to-one,

ii) Given a family $\{f_i\}_{i \in I} \subset \mathscr{C}(X)$ such that $\sup_{i \in I} f_i$ belongs to $\mathscr{C}(X)$ (i.e., the supremum is not identically equal to $+\infty$), it holds that $T(\sup_{i \in I} f_i) = \sup_{i \in I} T(f_i)$.

An important consequence of Proposition 1(ii) is that an operator in $\mathscr{B}$ is fully determined by its action on the family of affine functions on X, because any element of $\mathscr{C}(X)$ is indeed a supremum of affine functions (see, e.g., [19], p. 90). Hence, we will analyze next the behavior of the restrictions of operators in $\mathscr{B}$ to the family of affine functions.

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Define $\mathscr{A} \subset \mathscr{C}(X)$ as $\mathscr{A} = \{h : X \to \mathbb{R}$ such that h is affine and continuous$\}$. For $u \in X^*, \alpha \in \mathbb{R}$, define $h_{u,\alpha} \in \mathscr{A}$ as $h_{u,\alpha}(x) = \langle u, x\rangle + \alpha$. Clearly, every $h \in \mathscr{A}$ is of the form $h = h_{u,\alpha}$ for some $(u, \alpha) \in X^* \times \mathbb{R}$. For $f \in \mathscr{C}(X)$, define $A(f) \subset \mathscr{A}$ as $A(f) = \{h \in \mathscr{A} : h \leq f\}$.

We continue with an elementary characterization of affine functions.

Proposition 2. *For all $f \in \mathscr{C}(X)$, f belongs to $\mathscr{A}$ if and only if there exists a unique $u \in X^*$ such that $h_{u,\alpha} \in A(f)$ for some $\alpha \in \mathbb{R}$.*

Corollary 1. *If $f = h_{u,\alpha} \in \mathscr{A}$ then $A(f) = \{h_{u,\delta} : \delta \leq \alpha\}$.*

Corollary 2. *Consider $h_{u,\alpha} \in \mathscr{A}$, $f \in \mathscr{C}(X)$. If $f \leq h_{u,\alpha}$, then $f = h_{u,\delta}$ with $\delta \leq \alpha$.*

Next, we prove that operators in $\mathscr{B}$ map affine functions to affine functions.

Proposition 3. *If T belongs to $\mathscr{B}$ then*

i) $T(h) \in \mathscr{A}$ for all $h \in \mathscr{A}$,

ii) if $T(f) \in \mathscr{A}$ for some $f \in \mathscr{C}(X)$ then $f \in \mathscr{A}$.

Take $X = \mathbb{R}$ and define $\widehat{R} : \mathscr{A} \to \mathscr{A}$ as

$$\widehat{R}(h_{u,\alpha}) = \begin{cases} h_{u,\alpha} & \text{if } u \in (-1, 1) \\ h_{-u,\alpha} & \text{otherwise.} \end{cases}$$

We will prove next that $\widehat{T} : \mathscr{A} \to \mathscr{A}$ is indeed affine for all $T \in \mathscr{B}$, with the following meaning: we identify $\mathscr{A}$ with $X^* \times \mathbb{R}$, associating $h_{u,\alpha}$ to the pair (u, α), and hence we write $\widehat{T}(u, \alpha)$ instead of $\widehat{T}(h_{u,\alpha})$, and consider affinity of $\widehat{T}$ as an operator acting on $X^* \times \mathbb{R}$.

In view of Proposition 3, with this notation, for $T \in \mathscr{B}$, $\widehat{T}$ maps $X^* \times \mathbb{R}$ to $X^* \times \mathbb{R}$, and so we will write

$$\widehat{T}(u, \alpha) = (y(u, \alpha), \gamma(u, \alpha)), \tag{1}$$

with $y : X^* \times \mathbb{R} \to X^*, \gamma : X^* \times \mathbb{R} \to \mathbb{R}$, i.e. $y(u, \alpha)$ is the linear part of $T(u, \alpha)$ and $\gamma(u, \alpha)$ is its additive constant, so that $T(h_{u,\alpha})(x) = \langle y(u, \alpha), x\rangle + \gamma(u, \alpha)$.

We must prove that both y and γ are affine. We start with y, establishing some of its properties in the following two propositions.

Proposition 4. *If $T \in \mathscr{B}$ then $y(\cdot, \cdot)$, defined by (1), depends only upon its first argument.*

In view of Proposition 4, we will write in the sequel $y : X^* \to X^*$, with $y(u) = y(u, \alpha)$ for an arbitrary $\alpha \in \mathbb{R}$, and also

$$\widehat{T}(u, \alpha) = (y(u), \gamma(u, \alpha)). \tag{2}$$

We recall that $\phi : X^* \to \mathbb{R}$ is *quasiconvex* if $\phi(su_1 + (1 - s)u_2) \leq \max\{\phi(u_1), \phi(u_2)\}$ for all $u_1, u_2 \in X^*$ and all $s \in [0, 1]$.

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Proposition 5. *Assume that T belongs to $\mathscr{B}$. Then,*

i) $y : X^* \to X^*$, defined by (2), is one-to-one and onto,

ii) both $\langle y(\cdot), x\rangle : X^* \to \mathbb{R}$ and $\langle y^{-1}(\cdot), x\rangle : X^* \to \mathbb{R}$ are quasiconvex for all $x \in X$.

Next we recall a well known result on finite dimensional affine geometry.

Lemma 1. *Let V, V' be finite dimensional real vector spaces, with $\dim(V) \geq 2$. If $Q : V \to V'$ is invertible and maps segments to segments, then Q is affine.*

The next corollary extends this result to our infinite dimensional setting.

Corollary 3. *Let Z, Z' be arbitrary real vector spaces with $\dim(Z) \geq 2$. If $Q : Z \to Z'$ is invertible and maps segments to segments, then Q is affine.*

We continue with another result, possibly of some interest on its own.

Lemma 2. *If a mapping $M : X^* \to X^*$ satisfies:*

i) M is one-to-one and onto,

ii) both $\langle M(\cdot), x\rangle : X^* \to \mathbb{R}$ and $\langle M^{-1}(\cdot), x\rangle : X^* \to \mathbb{R}$ are quasiconvex for all $x \in X$,

then M is affine.

Proposition 6. *If T belongs to $\mathscr{B}$ then $\gamma(u, \cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing, one-to-one and and onto for all $u \in X^*$.*

Next we establish affinity of $\gamma(\cdot, \cdot)$.

Proposition 7. *If $T \in \mathscr{B}$ then $\gamma : X^* \times \mathbb{R} \to \mathbb{R}$, defined by (2), is affine.*

In the finite dimensional case, Proposition 7 would suffice for obtaining an explicit form of $\widehat{T}$, but in our setting we still have to prove continuity of $\widehat{T}$, which is not immediate from its affinity. For proving continuity, we will need the following elementary result.

Proposition 8. *Take $T \in \mathscr{B}$. If $f \in \mathscr{C}(X)$ is finite everywhere, then $T(f)$ is also finite everywhere.*

Proposition 9. *If $T \in \mathscr{B}$ then both $y : X^* \to X^*$ and $\gamma : X^* \times \mathbb{R} \to \mathbb{R}$ are continuous.*

Now we present a more explicit formula for the operator $\widehat{T}$.

Corollary 4. *If $T \in \mathscr{B}$, then there exist $d \in X^{**}$, $w \in X^*$, $\beta \in \mathbb{R}$, $\tau \in \mathbb{R}_{++}$ and a continuous automorphism D of X^* such that $\widehat{T}(u, \alpha) = (Du + w, \langle d, u\rangle + \tau\alpha + \beta)$.*

We recall now two basic properties of the Fenchel biconjugate, which will be needed in the proof of our main result.

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Proposition 10. *i) For all $f \in \mathscr{C}(X), f^{**}_{|X} = f$, where $f^{**}_{|X}$ denotes the restriction of f^{**} to $X \subset X^{**}$.*

ii) For all $f \in \mathscr{C}(X)$ and all $a \in X^{**}$, $f^{**}(a) = \sup_{h_{u,\alpha} \in A(f)} \{\langle a, u\rangle + \alpha\}$.

We mention, parenthetically, that the Fenchel-Moreau Theorem can be easily deduced from Proposition 10(ii). Next we present our first main result, characterizing the operators $T \in \mathscr{B}$

Theorem 1. *An operator $T : \mathscr{C}(X) \to \mathscr{C}(X)$ is fully order preserving if and only if there exist $c \in X$, $w \in X^*$, $\beta \in \mathbb{R}$, $\tau \in \mathbb{R}_{++}$ and a continuous automorphism E of X such that*

$$T(f)(x) = \tau f(Ex + c) + \langle w, x\rangle + \beta, \tag{3}$$

for all $f \in \mathscr{C}(X)$ and all $x \in X$.

The result in Theorem 1 can be rephrased as saying that the identity operator is the only fully order preserving operator in $\mathscr{C}(X)$, up to addition of affine functionals, pre-composition with affine operators, and multiplication by positive scalars, thus extending to Banach spaces the result established in [4] for the finite dimensional case.

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