On the Poincaré map and limit cycles for a class of continuous piecewise linear differential systems with three zones

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Resumo: In this work, we are interested in study the existence of limit cycles for the class of continuous piecewise linear differential systems with three zones

\[ \mathbf{x}' = X(\mathbf{x}), \]  

where \( \mathbf{x} = (x, y) \in \mathbb{R}^2 \), and \( X \) is a continuous piecewise linear vector field. For this, we make a thorough analysis of the Poincaré map for such vector fields.

Palavras-chave: piecewise vector field, limit cycles, Poincaré map.

1 Introduction

Due to the encouraging increase in their applications, control theory [5] and [9], design of electric circuits [2], neurobiology [4] and [8] piecewise linear differential systems were studied early from the point of view of qualitative theory of ordinary differential equations [1]. Nowadays, a lot of papers are being devoted to these differential systems.

On the other hand, starting from linear theory, in order to capture nonlinear phenomena, a natural step is to consider piecewise linear systems. As local linearizations are widely used to study local behavior, global linearizations (achieved quite naturally by working with models which are piecewise linear) can help to understand the richness of complex phenomena observed in the nonlinear world.

The study of piecewise linear systems can be a difficult task that is not within the scope of traditional nonlinear systems analysis techniques. In particular, a sound bifurcation theory is lacking for such systems due to their nonsmooth character.

In this work, we study the existence of limit cycles for the class of continuous piecewise linear differential systems

\[ \mathbf{x}' = X(\mathbf{x}), \]  

where \( \mathbf{x} = (x, y) \in \mathbb{R}^2 \), and \( X \) is a continuous piecewise linear vector field.

We will consider the following situation, that we will name the three-zone case. We have two parallel straight lines \( L_- \) and \( L_+ \) symmetric with respect to the origin dividing the phase plane in three closed regions: \( R_- \), \( R_0 \) and \( R_+ \) with \((0,0) \in R_0 \) and the regions \( R_- \) and \( R_+ \) have as boundary the straight lines \( L_- \) and \( L_+ \) respectively. We will denote by \( X_- \) the vector field \( X \) restricted to \( R_- \), by \( X_0 \) the vector field \( X \) restricted to \( R_0 \) and by \( X_+ \) the vector field \( X \) restricted to \( R_+ \). We suppose that the restriction of the vector field to each one of these zones are linear systems with constant coefficients that are glued continuously at the common boundary.
In short, system (2) can be written as

\[
x' = \begin{cases} 
A_-x + B_- & x \in R_-,
A_o x + B_o & x \in R_o,
A_+ x + B_+ & x \in R_+,
\end{cases}
\]

where \( A_i \in M_2(\mathbb{R}) \), \( i \in \{-, o, +\} \), \( B_i \in \mathbb{R}^2 \), \( i \in \{-, o, +\} \) and \( x' = \frac{dx}{dt} \) with \( t \) the time.

We say that the vector field \( X_i \) has a real equilibrium \( x^* \) in \( R_i \) with \( i \in \{-, o, +\} \) if \( x^* \) is an equilibrium of \( X_i \) and \( x^* \in R_i \). In opposite we will say that \( X_i \) has a virtual equilibrium \( x^* \) in \( R_i \) if \( x^* \) is an equilibrium of \( X_i \) and \( x^* \in R_i^c \), where \( R_i^c \) denotes the complementary of \( R_i \) in \( \mathbb{R}^2 \).

We suppose the following assumptions:

(H1) \( \mathcal{X}_o \) has an equilibrium of focus type.

(H2) The others equilibria, of \( \mathcal{X}_- \) and \( \mathcal{X}_+ \), are a center and a focus with different stability with respect to the focus of \( \mathcal{X}_o \).

We note that if the two equilibria of \( \mathcal{X}_- \) and \( \mathcal{X}_+ \) are both centers, or a center and a focus having this focus the same stability than the focus of \( \mathcal{X}_o \), then the vector field \( X \) has no limit cycles, for more details see for instance [3].

We denote \( t_i \) the trace of matrix \( A_i \), and by \( d_i \) the determinant of the matrix \( A_i \), for \( i \in \{-, o, +\} \).

**Proposition 1.** If system (4) has a simple invariant closed curve \( \Gamma \) then

\[
\int \int_{\text{Int}_o(\Gamma)} t_o \, dx \, dy + \int \int_{\text{Int}_{+}(\Gamma)} t_+ \, dx \, dy + \int \int_{\text{Int}_{-}(\Gamma)} t_- \, dx \, dy = t_- S_- + t_o S_o + t_+ S_+ = 0,
\]

where \( \text{Int}(\Gamma) \) is the open region limited by the closed Jordan curve \( \Gamma \), \( \text{Int}_i(\Gamma) = \text{Int}(\Gamma) \cap R_i \) and \( S_i = \text{area}(\text{Int}_i(\Gamma)) \) with \( i \in \{-, o, +\} \).

As usual a limit cycle of (3) is a periodic orbit of (3) isolated in the set of all periodic orbits of (3). A limit cycle is hyperbolic if the integral of the divergent of the system along it is different from zero, for more details see for instance [3].

## 2 Normal Form

The next result give us system (3) in a convenient normal form where the number of parameters are reduced, and consequently the computations of the Poincaré return map will be easier.

**Lemma 2.** Suppose that system (3) is such that \( \text{det}(A_o) > 0 \). Then there exists a linear change of coordinates that writes system (3) into the form

\[
\dot{x} = X(x),
\]

with \( L_- = L_{-1} = \{x = -1\} \), \( L_+ = L_1 = \{x = 1\} \), \( R_- = \{(x, y) \in \mathbb{R}^2; x \leq -1\} \), \( R_o = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1\} \), \( R_+ = \{(x, y) \in \mathbb{R}^2; x \geq 1\} \) and

\[
X(x) = \begin{cases} 
A_- x + B_- & x \in R_-,
A_o x + B_o & x \in R_o,
A_+ x + B_+ & x \in R_+,
\end{cases}
\]
where \( A_+ = \begin{pmatrix} a_{11} & -1 \\ 1 - b_2 + d_2 & a_1 \end{pmatrix}, \ B_+ = \begin{pmatrix} a_{11} & 0 \\ d_2 & a_1 \end{pmatrix}, \ A_o = \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}, \ B_o = \begin{pmatrix} 0 & 0 \\ b_2 & 1 \end{pmatrix}, \ A_- = \begin{pmatrix} c_{11} & -1 \\ 1 + b_2 - f_2 & a_1 \end{pmatrix} \) and \( B_- = \begin{pmatrix} -c_{11} \\ f_2 \end{pmatrix}. \) The dot denotes derivative with respect to a new time \( s. \)

Proof. By means of a rotation and a homothecy in the \( x \) direction we can write the system in such a way that \( L_- = L_1, \ L_+ = L_1. \) Now doing the change of coordinates given by \( u = x, \ v = k_1 x + k_2 y + k_3 \) and the time rescaling \( t = k_4 s \) with

\[
k_1 = -\frac{b_{11}}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}}, \quad k_2 = -\frac{b_{12}}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},
\]

\[
k_3 = -\frac{e_1}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}}, \quad k_4 = \frac{1}{\sqrt{b_{11}b_{22} - b_{12}b_{21}}},
\]

where \( b_{ij}, \ i, j = 1, 2 \) are the initial entries of the matrix \( A_o \) and \( B_o = (e_1, e_2), \) we have (4). \( \square \)

3 Poincaré map

For our purpose we will define a first return map that involves all the vector fields \( X_-, \ X_o \) and \( X_+ \) in a suitable transversal section.

In [6] and [7], a necessary and sufficient condition for the existence of Poincaré maps from the straight lines \( L_{\pm} \) to the straight lines \( L_{\pm} \) is that there exists a unique contact point of the flow of the linear system with these lines. By contact point we mean a point of the line where the vector field is tangent to it.

Lemma 3. In the coordinates given by Lemma 2 there is a unique contact point of system \( (4) \) with \( L_- \) and a unique contact point of \( (4) \) with \( L_+ \). These points are respectively \( p_- = (-1,0) \) and \( p_+ = (1,0) \). Moreover under the assumptions \( (H1) \) and \( (H2) \), we have

(i) if \( b_2 < -1 \), the equilibriums points of \( X_- \) and \( X_0 \) are virtual and the equilibrium point of \( X_+ \) is real;

(ii) if \( b_2 = -1 \), \( X_0 \) and \( X_+ \) have an equilibrium point at \( (1,0) \), and \( X_- \) has a virtual equilibrium point;

(iii) if \( |b_2| < 1 \), the equilibriums points of \( X_- \) and \( X_+ \) are virtual and the equilibrium point of \( X_0 \) is real;

(iv) if \( b_2 = 1 \), \( X_- \) and \( X_0 \) have an equilibrium point at \( (-1,0) \), and \( X_+ \) has a virtual equilibrium point;

(v) if \( b_2 > 1 \), the equilibriums points of \( X_0 \) and \( X_+ \) are virtual and the equilibrium point of \( X_- \) is real.

In the rest, we are considering \( b_2 < -1 \).

The Poincaré return map will be defined as the composition of four different Poincaré maps. In what follows we study the qualitative behavior of each one of these maps separately in order to understand the global behavior of the general Poincaré return map.

Let \( p_- \) be the contact point of \( \dot{x} = A_-x + B_- \) with \( L_- \). Note that \( p_- \) divides \( L_- \) into two segments \( L_-^o \) and \( L_-^l \) where in \( L_-^o \) the vector field points toward the region \( R_- \) while in \( L_-^l \) the vector field points toward the region \( R_o \). In fact we have \( L_-^o = \{ (-1,y); \ y \geq 0 \} \) and \( L_-^l = \{ (-1,y); \ y \leq 0 \} \).

We can define a Poincaré map \( \Pi_- : L_-^o \to L_-^l \) by \( \Pi_- (p) = q \) as the first return map in forward time of the flow of \( \dot{x} = A_-x + B_- \) to \( L_- \), that is, if \( \varphi_-(s,p) \) is the solution of \( \dot{x} = A_-x + B_- \)
such that \( \varphi_-(0, p) = p \) and \( p \in L^0_+ \), then \( q = \varphi_-(s, p), s \geq 0 \) such that \( q \in L^0_- \). Observe that \( \Pi_-(p_-) = p_- \).

We can see the mapping \( \Pi_- \) in a different way as follows. Given, \( p \in L^0_+ \) and \( q \in L^0_- \) there exist unique \( a \geq 0 \) and \( b \geq 0 \) such that \( p = p_- - a\dot{p}_- \) where \( \dot{p}_- = X_-(p_-) = (0, b_2 - 1) \), and \( q = p_- + b\dot{p}_- \). So the mapping \( \Pi_- \) induces a mapping \( \pi_- \) given by \( \pi_-(a) = b \).

Note that to study the qualitative behavior of \( \Pi_- \) is equivalent to study the qualitative behavior of \( \pi_- \). From now on we will consider the map \( \pi_- \) instead of \( \Pi_- \).

Note that we can define in the same way a map \( \pi_+ \) associated to the a Poincaré map \( \Pi_+ \) in \( L_+ \) considering the flow defined by \( \dot{x} = A_+x + B_+ \) and the contact point \( p_+ \).

### 3.1 Different types of Poincaré maps

In what follows, we present the study of Poincaré return maps defined in each zone. We denote the matrix \( A_+ \) by \( A_+ = \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & \alpha_+ \end{pmatrix} \), \( \tau_+ = \beta_+s \) and \( \gamma_+ = \alpha_+/\beta_+ \).

**Proposition 4.** Consider the vector field \( X_+ \) in \( R^2 \) with a virtual center. Let \( \pi_- \) be the map associated to the Poincaré map \( \Pi_- : L_- \to L_- \) defined by the flow of the linear system \( \dot{x} = A_-x + B_- \), then \( \pi_-(a) = -a \).

**Proposition 5.** Consider the vector field \( X_+ \) in \( R^2_+ \) with a real focus equilibrium and such that \( t_+ > 0 \). Let \( \pi_+ \) be the map associated to the Poincaré map \( \Pi_+ : L_+ \to L_+ \) defined by the flow of the linear system \( \dot{x} = A_+x + B_+ \).

(a) If \( t_+ > 0 \) then the maps \( \pi_+ \) satisfy that \( \pi_+ : [0, \infty) \to [b^*, \infty) \), \( \pi_+(0) = b^* > 0 \), \( \lim_{a \to \infty} \pi_+(a) = +\infty \) and \( \pi_+(a) > a \) in \((0, \infty)\).

(a.1) If \( a \in (0, \infty) \) then \( (\pi_+)'(a) = \frac{a}{\pi_+(a)}e^{2\gamma_+\tau_+} \). Moreover \( (\pi_+)'(a) > 0 \) and \( \lim_{a \to 0} (\pi_+)'(a) = 0 \).

(a.2) If \( a \in (0, \infty) \) then \( (\pi_+)'(a) > 0 \).

(a.3) The straight line \( b = e^{\gamma_+\tau_+}a + t_+(1 + e^{\gamma_+\pi_+})/d_+ \) in the plane \((a, b)\) is an asymptote of the graph of \( \pi_+ \) when \( a \) tends to \( +\infty \) where \( \gamma_+ = t_+ / \sqrt{4d_+ - t_+^2} \). So \( \lim_{a \to \infty} (\pi_+)'(a) = e^{\gamma_+} \).

**Proof.** Let \( p_+ \) be the contact point of the flow with \( L_+ \) and \( p \) and \( q \) as described above. As \( q \) is in the orbit of \( p \) in the forward time we have that \( q = \varphi(s, p) \) with \( s \geq 0 \). Moreover for computing the map \( \pi_+ \) we can suppose that the real equilibrium is at the origin and that matrix \( A_+ \) is in its real Jordan normal form.

Let \( p^*_+ \) be the contact point \( p_+ \) in the coordinates in which \( A_+ \) is in its real Jordan normal form and the virtual equilibrium of \( X_+ \) is at the origin. We denote by \( \dot{p}^*_+ = X_+(p^*_+) \). So we can write

\[
q = \varphi_+(s, p) = e^{A_+s}p.
\]

As \( p = p^*_+ + a\dot{p}^*_+ \) and \( q = p^*_+ - \pi_+(a)\dot{p}^*_+ \) we obtain

\[
p^*_+ - \pi_+(a)\dot{p}^*_+ = e^{A_+s}(p^*_+ + a\dot{p}^*_+).
\]

Now using the fact that \( \dot{p}^*_+ = A_+p^*_+ \) we have

\[
(Id - \pi_+(a)A_+)p^*_+ = e^{A_+s}(Id + aA_+)p^*_+,
\]

where \( a \geq 0 \), \( \pi_+(a) \geq 0 \), \( s \geq 0 \) and the matrix \( A_+ \) is given by \( A_+ = \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & \alpha_+ \end{pmatrix} \) with \( \alpha_+ = \frac{t_+}{2} \). Since \( p^*_+ \neq (0, 0) \) we obtain from equation (5) that \( b = \pi_+(a) \) is defined by the system

\[
1 - b\alpha_+ = e^{\alpha_+s}(\cos(\beta_+s) + a[\alpha_+ \cos(\beta_+s) - \beta_+ \sin(\beta_+s)]),
\]

\[
b\beta_+ = -e^{\alpha_+s}(\sin(\beta_+s) + a[\alpha_+ \sin(\beta_+s) + \beta_+ \cos(\beta_+s)]),
\]

where \( \alpha_+ = \frac{t_+}{2} \). Since \( p^*_+ \neq (0, 0) \) we obtain from equation (5) that \( b = \pi_+(a) \) is defined by the system

\[
1 - b\alpha_+ = e^{\alpha_+s}(\cos(\beta_+s) + a[\alpha_+ \cos(\beta_+s) - \beta_+ \sin(\beta_+s)]),
\]

\[
b\beta_+ = -e^{\alpha_+s}(\sin(\beta_+s) + a[\alpha_+ \sin(\beta_+s) + \beta_+ \cos(\beta_+s)]),
\]
and the inequalities \(a \geq 0\), \(b \geq 0\) and \(s \geq 0\).

Let be \(\pi_+(0) = b^*\), we have that \(b^* > 0\). Moreover if \(a = a_o\), \(b = b_o\) and \(s = s_o\) is a solution of (6), then \(s_o\) is the flight time between the points \(p = p^*_+ + a p^*_+ + q = p^*_+ - b p^*_+\).

Define \(\tau_+ = \beta_+ s\) and \(\gamma_+ = \alpha_+ / \beta_+\). Solving system (6) with respect to \(\tau_+\) we obtain the following parametric equations for \(\pi_+(a) = b\),

\[
a(\tau_+) = -\frac{\beta_+ e^{-\gamma_+ \tau_+}}{d_+ \sin \tau_+} \varphi(\tau_+, \gamma_+) \quad \text{and} \quad b(\tau_+) = -\frac{\beta_+ e^{\gamma_+ \tau_+}}{d_+ \sin \tau_+} \varphi(\tau_+, -\gamma_+),
\]

where \(\varphi\) is the function given by \(\varphi(x, y) = 1 - e^y (\cos x - y \sin x)\). Since \(A\) is given in its real Jordan normal form, \(\tau_+\) is the angle covered by the solution during the flight time \(s\). Hence we conclude that \(\tau_+ \in (\pi, \tau^*)\), where \(\tau^* < 2\pi\). Note that \(\tau^*\) is the angle covered by the solution during the flight time \(s^*\), i.e. \(e^{s^*} A p^*_+ = \Pi_+(p^*_+)\).

Now since \(\lim_{a \to \infty} a(\tau_+) = +\infty\) and \(\lim_{\tau_+ \to \pi^*} b(\tau_+) = +\infty\) it follows that the domain of definition of \(\pi_+\) is \([0, +\infty)\) and \(\lim_{a \to \infty} \pi_+(a) = +\infty\).

Moreover when \(\tau_+ \in (\pi, \tau^*)\) we have

\[
b(\tau_+) - a(\tau_+) = -\frac{2\beta_+}{d_+ \sin \tau_+} (\sinh(\gamma_+ \tau_+) - \gamma_+ \sin \tau_+).
\]

Since \(\sinh(\gamma_+ \tau_+) > \gamma_+ \sin \tau_+\) when \(\tau_+ \in (\pi, \tau^*)\), we conclude from the expression above that \(b(\tau_+) \geq a(\tau_+)\) if \(\tau_+ \in (\pi, \tau^*)\). Therefore \(\pi_+(a) > a\) in \((0, +\infty)\). So statement (a) is proved.

Derivating (7) with respect to \(\tau_+\) it follows that

\[
\frac{da}{d\tau_+} = -\frac{\beta_+}{d_+ \sin^2 \tau_+} \varphi(\tau_+, -\gamma_+) \quad \text{and} \quad \frac{db}{d\tau_+} = -\frac{\beta_+}{d_+ \sin^2 \tau_+} \varphi(\tau_+, \gamma_+).
\]

Thus \((\pi_+)'(a) = \frac{\varphi(\tau_+, \gamma_+)}{\varphi(\tau_+, -\gamma_+)} \frac{a}{b} e^{\gamma_+ \tau_+}\) and \(\lim_{a \to 0} (\pi_+)'(a) = 0\), because \(\lim_{a \to 0} b = \lim_{a \to 0} \pi_+(a) = b^* \neq 0\). Therefore substatement (a.1) is proved.

Now we observe that

\[
(\pi_+)'(a) = \frac{d}{d\tau_+} \left( \frac{db}{da} \right) \frac{1}{d \tau_+} \frac{da}{ds} = -\frac{2d_+ (1 + \gamma^2_+) \sin^3 \tau_+}{\beta_+ \varphi(\tau_+, -\gamma_+)^3} (\sinh(\gamma_+ \tau_+) - \gamma_+ \sin \tau_+) > 0.
\]

Therefore substatement (a.2) follows.

From expression (7) it follows that

\[
\lim_{a \to \infty} \frac{\pi_+(a)}{a} = \lim_{\tau_+ \to \pi} \frac{b(\tau_+)}{a(\tau_+)} = \lim_{\tau_+ \to \pi} \frac{e^{2\gamma_+ \tau_+} \varphi(\tau_+, -\gamma_+)}{\varphi(\tau_+, \gamma_+)} = e^{\gamma_+ \pi}.
\]

On the other hand by applying the L'Hôpital's rule it is easy to check that \(\lim_{a \to +\infty} (\pi_+(a) - e^{\gamma_+ \pi} a) = t (1 + e^{\gamma_+ \pi}) / d\), which implies that the straight line \(b = e^{\gamma_+ \pi} a + t (1 + e^{\gamma_+ \pi}) / d\) is an asymptote of the graph of \(\pi_+(a)\), we obtain substatement (a.3).

\[\square\]

**Proposition 6.** Consider the vector field \(X_+\) in \(R_+\) with a real focus equilibrium and such that \(t_+ < 0\). Let \(\pi_+\) be the map associated to the Poincaré map \(\Pi_+: L_+ \to L_+\) defined by the flow of the linear system \(\dot{x} = A_+ x + B_+\).
Consider the vector field $X: \mathbb{R}^2 \to \mathbb{R}^2$. We will give a sketch of proof.

**Proposition 7.** Consider the vector field $X_o$ in $R_o$ with a virtual focus equilibrium. Let $\pi_o$ be the map associated to the Poincaré map $\Pi_o: D_o^+ \subset L_+ \to L_+$ defined by the flow of the linear system $\dot{x} = A_o x + B_o$ from the straight line $L_+$ to the straight line $L_-

(a) Then the map $\pi_o$ satisfies that $\pi_o : [0, \infty) \to [c^*, \infty), c^* > 0$ with $\pi_o(0) = c^*$ and $\lim_{b \to -\infty} \pi_o(b) = +\infty$.

(b) If $b \in [0, \infty)$ then $\pi_o'(b) = (1 - b^2)^2 \frac{e^{2\pi_o \alpha}}{\pi_o(b)}$, with $\pi_o \to 0$ when $b \to \infty$ and $\lim_{b \to -\infty} \pi_o'(b) = b^2 - 1$.

**Proposition 8.** Consider the vector field $X_o$ in $R_o$ with a virtual focus equilibrium. Let $\bar{\pi}_o$ be the map associated to the Poincaré map $\bar{\Pi}_o: D_o^+ \subset L_+ \to L_-$ defined by the flow of the linear system $\dot{x} = A_o x + B_o$ from the straight line $L_+$ to the straight line $L_-$.

(a) Then the map $\bar{\pi}_o$ satisfies that $\bar{\pi}_o : [d^*, \infty) \to [0, \infty), d^* > 0$ with $\bar{\pi}_o(d^*) = 0$ and $\lim_{d \to -\infty} \bar{\pi}_o(d) = +\infty$.

(b) If $d \in (d^*, \infty)$ then $\bar{\pi}_o'(d) = \bar{\pi}_o'(d) = (b_2 + 1)^2 \frac{e^{2\pi_o \alpha}}{\bar{\pi}_o(d)}$, with $\pi_o \to 0$ when $d \to \infty$ and $\lim_{d \to -\infty} \bar{\pi}_o'(d) = b_2 + 1$ and $\lim_{d \to -d} \bar{\pi}_o'(d) = \infty$.

4 Limit cycles when $X_o$ has a virtual focus in $R_+$

In what follows without loss of generality we will suppose that the focus of hypothesis (H2) is in $R_+$.

**Proposition 9.** Assume that system (4) satisfies assumptions (H1) and (H2) and $b_2 < -1$. Then there exists a unique limit cycle of (4), which is hyperbolic. Moreover this limit cycle visits the three regions $R_-, R_o$ and $R_+$. It is a repeller if $t_o < 0$, and an attractor if $t_o > 0$.

**Proof.** We will give a sketch of proof.

Suppose that $X_-$ has a virtual center and $X_o$ has a virtual focus.

Using the previous notation we have $\gamma_i = \frac{\alpha_i}{\beta_i}, i \in \{-, o, +\}$. So $\gamma_- = 0$ and $\gamma_o, \gamma_+ \neq 0$. The domain of the first return map $\Pi$ defined by

$$\Pi = \pi_o \circ \bar{\pi} \circ \bar{\pi}_o \circ \pi_+$$

is an interval of $\mathbb{R}^+$ that depends on the domain of the mapping $\pi_+$ and $\bar{\pi}_o$ stated in Propositions 5, 6 and 8.
Suppose that $\gamma_o > 0$ and $\gamma_+ < 0$. In this case $\check{D}_o(\check{\pi}_o) = [d^*, \infty)$ where $d^* \geq 0$ and $\check{\pi}_o(d^*) = 0$. This implies that $D(\Pi) = [k^*, \infty)$ where $k^* = \pi_{-1}^{-1}(d^*)$. Moreover we have $\Pi : [k^*, \infty) \rightarrow [c^*, \infty)$ where $\Pi(k^*) = c^* = \pi_o(0)$.

Define the displacement function

$$h(a) = \Pi(a) - a.$$ 

Note that finding a fixed point of $\Pi$ is equivalent to find zeroes of the function $h$.

The objective is to study the nature of the application $\Pi$ and shown that this application is monotone in its domain. For this, we use that $\Pi' = \pi'_o(\pi_\circ \check{\pi}_o \circ \pi_+) \cdot \pi_1'(\check{\pi}_o \circ \pi_+) \cdot \pi_2'(\pi_+) \cdot \pi_3'$ and the expressions of each of its derivatives and signals given by the propositions 4, 5, 6, 7 and 8. Therefore, we conclude that if $\Pi$ has a fixed point, it is unique.

\[\square\]

References


